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INDEX.

Poincaré

*Systems of Ternariants that are Algebraically Complete.**

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The present memoir is divided into three parts; it deals with the theory of the algebraically independent concomitants of ternary quantics, taking as the starting point the six linear partial differential equations of the first order satisfied by them.

In Part I it is proved by means of these differential equations that, if any concomitant be expanded in powers of x_1, x_2, x_3—the ordinary (point) variables—and of u_1, u_2, u_3—the contragredient (line) variables—it is completely determinate if its leading coefficient be known; i. e. the coefficient of the term involving the highest power of x_1 and the highest power of u_1; and that every such leading coefficient is a simultaneous solution of two linear partial differential equations of the first order.

Hence if all the independent solutions of these two equations be obtained, all the independent concomitants are given—the independence considered being algebraical and not asyzygetic. For it follows from the theory of such differential equations that every solution can be expressed in terms of a finite number of such solutions, and therefore, on account of the uniqueness of the concomitant derived from a leading coefficient, that every concomitant can be expressed in terms of a finite number of independent concomitants.

Again, it follows from the forms of the two characteristic differential equations that every leading coefficient is a simultaneous concomitant of certain binary quantics constructed from the original ternary quantic, two of the coeffi-

* It has proved convenient to use the word "ternariants" as a generic term for concomitants of ternary quantics, instead of giving it the signification which Prof. Sylvester, to whom so much of the nomenclature of the theory of forms is due, proposed (*Amer. Journ. of Math.*, Vol. V, p. 81) to give to it, viz. the leading coefficients of those concomitants.

cients of which are their variables and the rest of the coefficients of which are their coefficients. There is thus a corresponding theory of these simultaneous binariants, the forms of which are of the utmost importance in the deduction of the leading coefficients of ternariants.

In Part II the foregoing general theory is applied to a number of cases of uniternary quantics, the two characteristic equations being solved in each case to give the independent simultaneous solutions as leading coefficients. It appears that in the case of the quadratic there are three ternariants of the nature indicated (they are shown to be equivalent to the three asyzygetic ternariants in the ordinary theory of the quadratic); in the case of the cubic there are seven, and in the case of the quartic there are twelve such ternariants. The leading coefficients of each of these are explicitly given for the most general form of quantic; and by one or other of two methods, viz. by means of the differential operators which give the development of the quantic, or by obtaining the symbolical expression of the ternariant with a given leading coefficient, the order in the point-variables and the class in the line-variables are derived. The three essential elements of each ternariant—its source, its order, and its class—being thus known, the full development for tabulation purposes is then only a question of differential operators, or of the realization of symbolical expressions.

Some illustrations are given in connection with the cubic and the quartic, wherein some of the well-known irreducible concomitants are algebraically expressed in terms of members of the fundamental sets. There is thus afforded an opportunity of comparing the functions which are algebraically independent with those which are only asyzygetically independent. Speaking generally, the result of the comparison is that for algebraical independence the order and the class are both much higher than for asyzygetic independence, so that in the case of complete tabulation many more terms in variables would occur; but that the leading (and so all the other) coefficients are much simpler in form (that is, contain fewer terms and are of lower degrees) than the leading coefficients of the asyzygetic ternariants.

The theory is then applied to the unipartite ternary quantic of order n, with the result that its fundamental system contains $\frac{1}{2}(n+4)(n-1)$ algebraically independent concomitants in terms of which every concomitant can be expressed. The forms, sometimes explicit, sometimes symbolical in the notation of only binary quantics, are given for these leading coefficients; the order and

the class of each ternariant thus determined are obtained from the symbolical expressions.

The case of two simultaneous ternary quadratics is next discussed, and it appears that there are nine ternariants in the fundamental system. Some subsidiary ternariants (subsidiary, that is, from the point of view of the fundamental system) are obtained; and the 20 ternariants which constitute the asyzygetic system due to Gordan, are expressed in terms of the foregoing system.

Lastly, the fundamental system of three simultaneous ternary quadratics is obtained.

In Part III the general theory is applied to a number of cases of biternary quantics; the special cases treated at any length being the systems of the lineo-linear, the quadrato-linear, the cubo-linear, the quadrato-quadratic, the cubo-cubic, and the system of two lineo-linear quantics. Finally, the fundamental system of the biternary n^0m^{ic} is obtained, containing

$$\tfrac{1}{4}(n+1)(n+2)(m+1)(m+2)-3$$

ternariants, of which the greatest number are, for this most general case, given in symbolic form (the equivalent of solutions of the two characteristic equations), and for each of which the order and the class are derived from that symbolical form. In some cases of the last, ternariants syzygetically equivalent, save for a power of u_x, to members of the fundamental system, are given.

There is an immense mass of literature on the subject of ternary forms; but, so far as I can see, it deals with asyzygetic ternariants which are more difficult to treat, and have been given in full systems only for the uni-ternary quadratic, the uni-ternary cubic, and a system of two uni-ternary quadratics, for a special form of uni-ternary quartic and for a biternary lineo-linear quantic; and part of the system has been given for the general uni-ternary quartic.

Some of the principal papers dealing with the theory of the cubic are given in Cayley's memoir.* Gordan's memoirs on the special form of the ternary quartic occur in the 17th and 20th volumes of the Mathematische Annalen; the concomitants of lowest degrees for the general quartic are given in a memoir by Maisano (quoted in §42); and several of the concomitants are tabulated in full

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* *Amer. Journ. of Math.*, Vol. IV, 1881, On the 34 Concomitants of the Ternary Cubic, pp. 1–15.

in a memoir by Joubert.* In connection with the general theory, a memoir by Sylvester† may be studied; and also a recent memoir by Brioschi.‡ In a memoir by Bernardi,§ the characteristic differential equations satisfied by concomitants of uni-ternary concomitants are given, and the Hessian and the cubinvariant of the quartic are there calculated in full.

In regard to biternary forms in sets of contragredient variables, the most important memoir is by Clebsch and Gordan, quoted in the note to §60; in that note a fairly complete list of the more important papers dealing with bipartite forms is given, though most of them discuss the transformation of bilinear forms.||

PART I.

GENERAL THEORY.

The Differential Equations Satisfied by the Concomitants.

1. In the discussion of the concomitants of ternary quantics, the ordinary (point) variables will be denoted by x_1, x_2, x_3; the contragredient (line) variables by u_1, u_2, u_3. The general uni-ternary quantic of order n may be represented either in the symbolical form

$$f = a_x^n = (a_1x_1 + a_2x_2 + a_3x_3)^n,$$

or in the reduced non-symbolical form

$$f = (\ldots\ldots, a_{rst}, \ldots\ldots \between x_1, x_2, x_3)^n,$$

where the coefficient of $a_{rst}x_1^rx_2^sx_3^t$ is the multinomial coefficient $n! \div (r!\, s!\, t!)$ with the condition $r + s + t = n$.

Every concomitant ϕ, of general order m in the point variables and general class p in the line variables, satisfies** the six characteristic linear partial differential equations

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* "Sur la théorie algébrique des formes homogènes du quatrième degré à trois indéterminées." Comptes Rendus, t. LVI (1863), pp. 1045–1048, 1088–1091, 1123–1126.

† Comptes Rendus, t. LXXXIV (1877), pp. 1359–1361, 1427–1430.

‡ "Studi sulle forme ternarie," Ann. di Mat., t. XV (1888), pp. 235–252.

§ Batt. Giorn., t. XIX (1881), pp. 136–150, 258–293.

|| A short table of contents of this paper is given at the end.

** "On the Differential Equations satisfied by Concomitants of Quantics," Proc. Lond. Math. Soc., Vol. XIX (1888), p. 41.

$$\left.\begin{aligned}
x_1\frac{\partial\phi}{\partial x_2}-u_2\frac{\partial\phi}{\partial u_1}&=\Sigma\Sigma r a_{r-1,\,s+1,\,t}\frac{\partial\phi}{\partial a_{rst}}=D_3\phi\\
x_2\frac{\partial\phi}{\partial x_3}-u_3\frac{\partial\phi}{\partial u_2}&=\Sigma\Sigma s a_{r,\,s-1,\,t+1}\frac{\partial\phi}{\partial a_{rst}}=D_1\phi\\
x_3\frac{\partial\phi}{\partial x_1}-u_1\frac{\partial\phi}{\partial u_3}&=\Sigma\Sigma t a_{r+1,\,s,\,t-1}\frac{\partial\phi}{\partial a_{rst}}=D_2\phi\\
x_2\frac{\partial\phi}{\partial x_1}-u_1\frac{\partial\phi}{\partial u_2}&=\Sigma\Sigma s a_{r+1,\,s-1,\,t}\frac{\partial\phi}{\partial a_{rst}}=D_4\phi\\
x_3\frac{\partial\phi}{\partial x_2}-u_2\frac{\partial\phi}{\partial u_3}&=\Sigma\Sigma t a_{r,\,s+1,\,t-1}\frac{\partial\phi}{\partial a_{rst}}=D_6\phi\\
x_1\frac{\partial\phi}{\partial x_3}-u_3\frac{\partial\phi}{\partial u_1}&=\Sigma\Sigma r a_{r-1,\,s,\,t+1}\frac{\partial\phi}{\partial a_{rst}}=D_5\phi
\end{aligned}\right\}\quad(1),$$

where the summations in the literal operators extend to all distinct integral combinations of r, s, t subject to the relation $r+s+t=n$.

2. The 15 Jacobian conditions, which must be satisfied in order that these equations may have common solutions, resolve themselves into three classes. The first class consists of 6 equations satisfied identically, in virtue of the relations

$$[D_3,D_6]=0,\quad [D_1,D_5]=0,\quad [D_2,D_4]=0,$$
$$[D_3,D_5]=0,\quad [D_1,D_4]=0,\quad [D_2,D_6]=0.$$

The second class consists of six equations satisfied by means of some one of the equations (1), in virtue of the relations

$$[D_1,D_2]=D_4,\quad [D_2,D_3]=D_6,\quad [D_3,D_1]=D_5,$$
$$[D_4,D_5]=D_1,\quad [D_5,D_6]=D_3,\quad [D_6,D_4]=D_2.$$

The third class consists of three quasi-homogeneous equations, viz.

$$\left.\begin{aligned}
D_9\phi=[D_3,D_4]\,\phi&=x_1\frac{\partial\phi}{\partial x_1}-x_2\frac{\partial\phi}{\partial x_2}-u_1\frac{\partial\phi}{\partial u_1}+u_2\frac{\partial\phi}{\partial u_2}\\
D_8\phi=[D_2,D_5]\,\phi&=x_3\frac{\partial\phi}{\partial x_3}-x_1\frac{\partial\phi}{\partial x_1}-u_3\frac{\partial\phi}{\partial u_3}+u_1\frac{\partial\phi}{\partial u_1}\\
D_7\phi=[D_1,D_6]\,\phi&=x_2\frac{\partial\phi}{\partial x_2}-x_3\frac{\partial\phi}{\partial x_3}-u_2\frac{\partial\phi}{\partial u_2}+u_3\frac{\partial\phi}{\partial u_3}
\end{aligned}\right\}\quad(2),$$

where the operators D_7, D_8, D_9 are

$$\left.\begin{aligned}
D_7&=\Sigma\Sigma\,(s-t)\,a_{rst}\frac{\partial}{\partial a_{rst}}\\
D_8&=\Sigma\Sigma\,(t-r)\,a_{rst}\frac{\partial}{\partial a_{rst}}\\
D_9&=\Sigma\Sigma\,(r-s)\,a_{rst}\frac{\partial}{\partial a_{rst}}
\end{aligned}\right\}\quad(3).$$

And the further Jacobian conditions, due to the introduction of the new equations (2), are found to be satisfied, in virtue of relations of the form

$$\begin{aligned} &[D_1, D_9] = D_1, \quad [D_5, D_9] = -D_5, \quad [D_3, D_9] = -2D_3, \\ &[D_2, D_9] = D_2, \quad [D_6, D_9] = -D_6, \quad [D_4, D_9] = 2D_4, \end{aligned}$$

with similar relations for the other two operators D_7 and D_8.

3. But the equations in the foregoing system, sufficient for the derivation of every partial differential equation satisfied by a concomitant, are not independent of one another.

Let any equation be written in the form $P'\phi = P\phi$, where P is the literal operator and P' is the variable operator; then any one of the nine operators P is commutative with any one of the nine operators P'. Consider, then, any function Φ which satisfies two of the equations, say

$$P'\phi = P\phi, \quad Q'\phi = Q\phi;$$

then we have

$$\begin{aligned} (PQ - QP)\phi &= (PQ' - QP')\phi, \\ &= (Q'P - P'Q)\phi, \\ &= (Q'P' - P'Q')\phi. \end{aligned}$$

If the operators P and Q be commutative, then the equation just obtained is an evanescent identity; but if they be not commutative, the equation is not an identity and yet it is different in form from either of the two from which it is derived. It is thus a new equation,

$$R\phi = R'\phi,$$

satisfied in virtue of the two former equations.

When this process is applied to the foregoing system of nine equations, it appears that they can be reduced to a system of three equations, independent of one another, by means of the following sets of relations:

$$\left.\begin{aligned} D'_1 &= D'_4D'_5 - D'_5D'_4 \\ D'_2 &= D'_6D'_4 - D'_4D'_6 \\ D'_3 &= D'_5D'_6 - D'_6D'_5 \\ D'_4 &= D'_1D'_2 - D'_2D'_1 \\ D'_5 &= D'_3D'_1 - D'_1D'_3 \\ D'_6 &= D'_2D'_3 - D'_3D'_2 \\ D'_7 &= D'_1D'_6 - D'_6D'_1 \\ D'_8 &= D'_2D'_5 - D'_5D'_2 \\ D'_9 &= D'_3D'_4 - D'_4D'_3 \end{aligned}\right\}; \qquad \left.\begin{aligned} D_1 &= D_5D_4 - D_4D_5 \\ D_2 &= D_4D_6 - D_6D_4 \\ D_3 &= D_6D_5 - D_5D_6 \\ D_4 &= D_2D_1 - D_1D_2 \\ D_5 &= D_1D_3 - D_3D_1 \\ D_6 &= D_3D_2 - D_2D_3 \\ D_7 &= D_6D_1 - D_1D_6 \\ D_8 &= D_5D_2 - D_2D_5 \\ D_9 &= D_4D_3 - D_3D_4 \end{aligned}\right\}.$$

Thus, for instance, the nine equations can be made to depend on the set

$$D_1\phi = D_1'\phi\,; \quad D_2\phi = D_2'\phi\,; \quad D_3\phi = D_3'\phi\,.$$

But it will be convenient to retain the whole system, for it includes the full aggregate of equations which are similar to one another in form.

All combinations, other than those which occur in the foregoing set of combinations, are connected with operators which are lineo-commutative; e. g. $D_1D_4 - D_4D_1 = 0$. And it may be remarked that the operators in the foregoing set of combinations are quadrato-commutative, according to laws of the form

$$D_2D_1^2 - 2D_1D_2D_1 + D_1^2D_2 = 0\,,$$
$$D_1D_2^2 - 2D_2D_1D_2 + D_2^2D_1 = 0\,.$$

4. It is convenient to assign certain weights to the various quantities which enter. We assign the weight zero to x_3, unity to x_2, and ρ (unspecified but, if desirable, different from zero or unity) to x_1. Since f and u_x $(= u_1x_1 + u_2x_2 + u_3x_3)$ must be isobaric, we assign to a_1 and u_1 the weight zero, to a_2 and u_2 the weight $\rho - 1$, and to a_3 and u_3 the weight ρ.* Then the weight of the coefficient $a_{r,\,s,\,t}$ is $s(\rho - 1) + t\rho$; and the following changes are caused on any isobaric function by the operators:

The operator D_1 increases and D_6 decreases the weight by 1,
" " D_3 " " D_4 " " " " $\rho - 1$,
" " D_5 " " D_2 " " " " ρ.

5. Now suppose the concomitant Φ expanded in powers of the point variables, in which its order is m; this expansion is of the form

$$\Phi = x_1^m\Phi_{0,\,0} + \ldots\ldots + \frac{x_1^{m-r-s}x_2^r x_3^s}{r!\,s!}\Phi_{r,\,s} + \ldots\ldots \qquad (4).$$

When this expression is substituted in the six fundamental characteristic equations (1), the result is in each case an identity; hence, by comparison of the coefficients of the various x-combinations, we have the relations

$$\left.\begin{aligned} D_3\Phi_{r,\,s} + u_2\frac{\partial\Phi_{r,\,s}}{\partial u_1} &= \Phi_{r+1,\,s} \\ D_5\Phi_{r,\,s} + u_3\frac{\partial\Phi_{r,\,s}}{\partial u_1} &= \Phi_{r,\,s+1} \end{aligned}\right\} \qquad (5);$$

*The weights σ, $\sigma + \rho - 1$, $\sigma + \rho$ (σ unspecified) might be assigned to u_1, u_2, u_3; but the foregoing assignation is equally efficient for the purpose of obtaining the difference in weights of the variable parts of two terms of a concomitant.

$$\left.\begin{aligned}
D_1\Phi_{r,s}+u_3\frac{\partial\Phi_{r,s}}{\partial u_2}&=r\Phi_{r-1,s+1}\\
D_6\Phi_{r,s}+u_2\frac{\partial\Phi_{r,s}}{\partial u_3}&=s\Phi_{r+1,s-1}\\
D_2\Phi_{r,s}+u_1\frac{\partial\Phi_{r,s}}{\partial u_3}&=(m-r-s+1)s\Phi_{r,s-1}\\
D_4\Phi_{r,s}+u_1\frac{\partial\Phi_{r,s}}{\partial u_2}&=(m-r-s+1)r\Phi_{r-1,s}
\end{aligned}\right\}.$$

From the first pair of these, viz. (5), it follows that if Φ_{00} and m be known, the full expansion of the covariant can be obtained merely by processes of differentiation, for the two equations give the relation

$$\Phi_{r,s}=\left(D_3+u_2\frac{\partial}{\partial u_1}\right)^r\left(D_5+u_3\frac{\partial}{\partial u_1}\right)^s\Phi_{0,0} \qquad \text{(I)}.$$

And from the remaining four it follows that $\Phi_{0,0}\,(=P)$ satisfies the four equations

$$\left.\begin{aligned}
D_1P+u_3\frac{\partial P}{\partial u_2}&=0, \quad D_2P+u_1\frac{\partial P}{\partial u_3}=0\\
D_6P+u_2\frac{\partial P}{\partial u_3}&=0, \quad D_4P+u_1\frac{\partial P}{\partial u_2}=0
\end{aligned}\right\} \qquad (6).$$

6. One immediate inference as to the isobaric character of a concomitant can be derived; for if P be assumed isobaric and of weight ε, then the weight of $\Phi_{r,s}$ is, on account of the effect of the operators D_3 and D_5, equal to $\varepsilon+r(\rho-1)+s\rho$, and therefore the weight of the term $x_1^{m-r-s}x_2^r x_3^s\Phi_{r,s}$ is

$$(m-r-s)\,\rho+r+\varepsilon+\varepsilon\,(\rho-1)+s\rho,$$

that is, it is $m\rho+\varepsilon$, and is therefore the same for every term.

7. Let $\Phi_{00}\,(=P)$, which in general is a function of u_1, u_2, u_3 of class p, be expanded in powers of these variables in the form

$$P=u_1^p\psi_{0,0}+\ldots\ldots+\frac{u_1^{p-q-t}u_2^q u_3^t}{q!\,t!}\psi_{q,t}+\ldots\ldots \qquad (7),$$

in which the quantities ψ involve only the coefficients of the quantic. Then proceeding as before, a comparison of the various combinations of the variables in the equations (6) after the substitution of P leads to the relations

$$\left.\begin{aligned}
D_1\psi_{q,t}+s\psi_{q+1,t-1}&=0\\
D_6\psi_{q,t}+r\psi_{q-1,t+1}&=0
\end{aligned}\right\}$$
$$\left.\begin{aligned}
D_2\psi_{q,t}+\psi_{q,t+1}&=0\\
D_4\psi_{q,t}+\psi_{q+1,t}&=0
\end{aligned}\right\} \qquad (8).$$

From the last pair of these, viz. (8), it follows that if $\psi_{0,0}$ be known and also p, the full expansion of P can be obtained merely by processes of differentiation; for the two equations give the relation

$$\psi_{q,t} = (-1)^{q+t} D_4^q D_2^t \psi_{0,0} \qquad \text{(II)}.$$

But this relation shows that if $\psi_{0,0}$ be known, the value of p can be derived from it. For the term involving the highest power of u_3 has, save as to a numerical factor, the coefficient $\psi_{0,p}$; and a coefficient $\psi_{0,p+1}$ is necessarily zero. Hence

and similarly
$$\left.\begin{aligned} D_2^{p+1}\psi_{0,0} &= 0 \\ D_4^{p+1}\psi_{0,0} &= 0 \end{aligned}\right\} \qquad (9).$$

The value of p can thus be obtained when $\psi_{0,0}$ is given by operating with D_2 or D_4 a number of times in succession; the value is evidently less by unity than the number of times first necessary to give a zero result.

From the former pair of equations it follows that $\psi_{0,0}$ satisfies the two equations

$$D_1\psi_{0,0} = 0, \quad D_6\psi_{0,0} = 0 \qquad (10).$$

8. Before proceeding to discuss the effect of the subsidiary characteristic equations (2), it is desirable to reconsider the main equations (1).

The preceding results have been deduced on the supposition that the concomitant Φ is most conveniently arranged initially in powers of the point-variables. But if we take an alternative initial expression in powers of the line-variables—a necessity in the case of pure contravariants—in the form

$$\Phi = u_1^p \chi_{0,0} + \ldots + (-1)^{r+s} \frac{u_1^{p-r-s} u_2^r u_3^s}{r!\, s!} \chi_{r,s} + \ldots \qquad (4'),$$

and substitute in the original equations, then similar analysis gives the two results: (i) that $\chi_{r,s}$ is determined from $\chi_{0,0}$ by the relation

$$\chi_{r,s} = \left(D_2 - x_3 \frac{\partial}{\partial x_1}\right)^r \left(D_4 - x_2 \frac{\partial}{\partial x_1}\right)^s \chi_{0,0} \qquad \text{(III)},$$

and (ii) that $\chi_{0,0}\,(= Q)$ satisfies the equations

$$\left.\begin{aligned} D_3 Q = x_1 \frac{\partial Q}{\partial x_2}, \quad D_1 Q = x_2 \frac{\partial Q}{\partial x_3} \\ D_5 Q = x_1 \frac{\partial Q}{\partial x_3}, \quad D_6 Q = x_3 \frac{\partial Q}{\partial x_2} \end{aligned}\right\} \qquad (6').$$

And when Q, a function of the point-variables, is expanded in the form

$$Q = x_1^m \theta_{0,0} + \ldots + \frac{x_1^{m-q-t} x_2^q x_3^t}{q!\,t!} \theta_{q,t} + \ldots \qquad (7')$$

(with the evident relation $\theta_{0,0} = \psi_{0,0}$, each being the coefficient of $x_1^m u_1^p$ in Φ), then, by means of the equations (6′) it follows that

$$\theta_{q,t} = D_3^q D_5^t \theta_{0,0} = D_3^q D_5^t \psi_{0,0} \qquad (\text{IV});$$

that m can be determined by either of the equations

$$\left.\begin{array}{l} D_3^{m+1}\psi_{0,0} = 0 \\ D_5^{m+1}\psi_{0,0} = 0 \end{array}\right\} \qquad (11),$$

and that $\theta_{0,0}\,(=\psi_{0,0})$ satisfies the same equations as before, viz.

$$D_1\psi_{0,0} = 0, \quad D_6\psi_{0,0} = 0 \qquad (10').$$

9. Hence it follows that a concomitant is uniquely determined by the coefficient of its leading term; for by (9) and (11) its order and class are deducible from that leading coefficient, and either by means of (I) and (II) or by means of (III) and (IV) the full expansion can be obtained.

10. The determination of a concomitant thus resolves itself into the determination of the leading coefficient of that concomitant. We have already seen that it must satisfy the two equations (10); there remains the consideration of the effect of the subsidiary equations (2) on the leading coefficient. When the expanded form of the concomitant Φ is substituted, then so far as $\Phi_{0,0}$ is concerned we have the relations

$$D_9\psi_{0,0} = (m-p)\,\psi_{0,0} = -D_8\psi_{0,0}; \quad D_7\psi_{0,0} = 0,$$

which, on account of the forms of the operators D_7, D_8, D_9 are equivalent to

$$\left.\begin{array}{l} D_7\psi_{0,0} = 0 \\ D_9\psi_{0,0} = (m-p)\,\psi_{0,0} \end{array}\right\} \qquad (12).$$

Hence it follows that $\psi_{0,0}$ satisfies the two characteristic equations (10) and the two quasi-homogeneous equations (12).

11. As in the corresponding theorem in binary quantics, it follows that every function, derived by the complete set of equations through an isobaric solution $\psi_{0,0}$ of the determining differential equations (10) and (12), is a concomitant.

12. Two remarks remain to be made: one is that the operators used in

deducing the expansion of the concomitant are, in pairs, applicable in any order, viz. $D_3 + u_3 \frac{\partial}{\partial u_1}$ and $D_5 + u_3 \frac{\partial}{\partial u_1}$, D_3 and D_5, $D_4 - x_2 \frac{\partial}{\partial x_1}$ and $D_2 - x_3 \frac{\partial}{\partial x_1}$, D_4 and D_2. The other is that the expansion has been derived in descending powers of x_1 and u_1, while it might equally well have been so derived in regard to x_2 and u_2, or x_3 and u_3; and thus the leading coefficients of concomitants (which are not invariants) will be specially associated with coefficients of terms involving high powers of x_1 in the quantic, as later coefficients will be specially associated with coefficients of terms involving high powers of x_2 and of x_3.

13. The general inferences as to the equations which determine the various classes of concomitants are as follow:

The leading coefficients ψ of all *mixed concomitants* (Zwischenformen) satisfy and are determined by the characteristic equations

$$D_1\psi = 0, \quad D_6\psi = 0,$$

and, as will appear subsequently, every adopted solution of these two equations identically satisfies the subsidiary equations

$$D_7\psi = 0, \quad D_9\psi = (m - p)\psi,$$

where m is the order of the mixed concomitant in the point-variables and p is its class in the line-variables. The integers m and p are determinable from equations (9) and (11); and the full development of the concomitant is given by equations (I) and (II), or by (III) and (IV).

The leading coefficients of all *pure contravariants* (Zugehörige Formen) satisfy the characteristic equations

$$D_1\psi = 0, \quad D_6\psi = 0, \quad D_3\psi = 0, \quad D_5\psi = 0;$$

they satisfy identically the subsidiary equations

$$D_7\psi = 0, \quad D_9\psi = -p\psi,$$

where p is the class of the contravariant.

The leading coefficients of all *pure covariants* satisfy the characteristic equations

$$D_1\psi = 0, \quad D_6\psi = 0, \quad D_2\psi = 0, \quad D_4\psi = 0;$$

they satisfy identically the subsidiary equations

$$D_7\psi = 0, \quad D_9\psi = m\psi,$$

where m is the order of the covariant.

All *invariants* satisfy the characteristic equations

$$D_1\psi=0,\quad D_6\psi=0,\quad D_2\psi=0,\quad D_4\psi=0,\quad D_3\psi=0,\quad D_5\psi=0;$$

and they satisfy identically the subsidiary equations

$$D_7\psi=0,\quad D_8\psi=0.$$

14. It thus appears that from the point of view of analytical derivation of the concomitants, it is in every case necessary to obtain the common solutions of the two equations $D_1\psi=0=D_6\psi$, and that from every common solution the corresponding order and class can be deduced. Whenever either the order or the class, or both the order and the class, may happen to be zero, two additional equations for each zero are satisfied by the common solution of $D_1\psi=0=D_6\psi$.

It is not, however, necessary to determine both m and p, the order and the class, of a mixed concomitant by the processes indicated. Since $x_1^m u_1^p\psi$ is a term of the concomitant, so also $\pm$ is $x_2^m u_2^p\psi'$, where ψ' is the value of ψ when corresponding coefficients of the quantic are interchanged by the substitution $x_1=X_2$, $x_2=X_1$, $x_3=X_3$. If, then, W be the weight of ψ and W' that of ψ', we have from the isobarism of the concomitant

$$m\rho+W=m+p(\rho-1)+W',$$

so that

$$m-p=\frac{W'-W}{\rho-1},$$

a quantity thus determinable by mere inspection. No further relation to determine m and p, other than an equivalent of this relation, can be obtained by such interchanges and substitutions. In using the relation, moreover, it is sufficient to obtain W and W' from any—the simplest—term in ψ.

In order, then, to determine by this method the quantities m and p to be necessarily associated with the concomitant determinable by a given solution ψ, the first step will be to determine the value of $m-p$; the second will naturally be to determine the smaller of the two quantities (should they be unequal) by the equations (9) or (11).

There is, however, another method of proceeding which is much more rapid for the determination of m and p, though less advantageous for tabulation purposes. It is a consequence of the theorem that every ternariant can be represented symbolically, that a leading coefficient is sufficient to determine the ternariant uniquely; if, therefore, a leading coefficient be given, the most rapid

method of obtaining an expression is to change that leading coefficient so that it may be constituted solely by the umbral elements of the original quantic or quantics, and then to complete, by means of the variables, the various factors of that umbral form according to the laws that govern symbolical expressions. Thus for instance an umbral factor a_1 would be completed into a_x, an umbral factor $b_2c_3 - b_3c_2$ into (bcu), and so on.

15. Further, since each of the characteristic equations is linear and partial of the first order, there will be for each quantic a definite number M of algebraically independent solutions of $D_1\psi = 0 = D_6\psi$; and it is a consequence of the theory of such equations that every solution can be expressed as a function of these M solutions. It has been seen that each such isobaric, homogeneous solution determines a concomitant, and therefore for every quantic there is a definite number M of concomitants algebraically independent of one another, such that any concomitant of that quantic can be expressed in terms of those M concomitants, and of the universal concomitant $u_x = u_1x_1 + u_2x_2 + u_3x_3$.

Such a system of concomitants is not unique; it may be replaced by an algebraically equivalent system, containing necessarily the same number of independent concomitants. And the independence is not merely syzygetic, it is an algebraical independence.

16. In the matter of notation, it is desirable to have the coefficients of the quantics so chosen as to render the analytical forms of the characteristic differential equations as simple as possible. Thus the ternary quadratic is taken in the form

$$\begin{aligned} & a_0x_1^2 + 2x_3a_1x_1 + x_3^2a_2; \\ + & 2b_0x_1x_2 + 2x_3b_1x_2 \\ + & c_0x_2^2; \end{aligned}$$

the ternary cubic in the form

$$\begin{aligned} & a_0x_1^3 + 3x_3a_1x_1^2 + 3x_3^2a_2x_1 + x_3^3a_3; \\ + & 3b_0x_1^2x_2 + 3x_3 2b_1x_1x_2 + 3x_3^2b_2x_2 \\ + & 3c_0x_1x_2^2 + 3x_3c_1x_2^2 \\ + & d_0x_2^3, \end{aligned}$$

and the arrangement of the coefficients for the general quantic can evidently be made to follow the same law.

With this notation the six literal operators in equations are: first, the two

the simultaneous solutions of which have to be obtained are

$$\left.\begin{aligned} D_1 &= a_1\frac{\partial}{\partial b_0}+a_2\frac{\partial}{\partial b_1}+a_3\frac{\partial}{\partial b_2}+\ldots+2\left(b_1\frac{\partial}{\partial c_0}+b_2\frac{\partial}{\partial c_1}+\ldots\right)+3\left(c_1\frac{\partial}{\partial d_0}+\ldots\right)\\ D_6 &= b_0\frac{\partial}{\partial a_1}+c_0\frac{\partial}{\partial b_1}+d_0\frac{\partial}{\partial c_1}+\ldots+2\left(b_1\frac{\partial}{\partial a_2}+c_1\frac{\partial}{\partial b_2}+\ldots\right)+3\left(b_2\frac{\partial}{\partial a_3}+\ldots\right)\end{aligned}\right\};$$

second, the two which serve to give the development of the concomitant in powers of x and to determine m are

$$\left.\begin{aligned} D_3 = nb_0\frac{\partial}{\partial a_0}&+(n-1)\,c_0\frac{\partial}{\partial b_0}+(n-2)\,d_0\frac{\partial}{\partial c_0}+\ldots\\ &+(n-1)\,b_1\frac{\partial}{\partial a_1}+(n-2)\,c_1\frac{\partial}{\partial b_1}+\ldots+(n-2)\,b_2\frac{\partial}{\partial a_2}+\ldots\\ D_5 = na_1\frac{\partial}{\partial a_0}&+(n-1)\,a_2\frac{\partial}{\partial a_1}+(n-2)\,a_3\frac{\partial}{\partial a_2}+\ldots\\ &+(n-1)\,b_1\frac{\partial}{\partial b_0}+(n-2)\,b_2\frac{\partial}{\partial b_1}+\ldots+(n-2)\,c_1\frac{\partial}{\partial c_0}+\ldots\end{aligned}\right\};$$

third, the two which serve to give the development of the concomitant in powers of u and to determine p are

$$\left.\begin{aligned} D_2 &= a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+3a_2\frac{\partial}{\partial a_3}+\ldots+b_0\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial b_2}+\ldots+c_0\frac{\partial}{\partial c_1}+\ldots\\ D_4 &= a_0\frac{\partial}{\partial b_0}+2b_0\frac{\partial}{\partial c_0}+3c_0\frac{\partial}{\partial d_0}+\ldots+a_1\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial c_1}+\ldots+a_2\frac{\partial}{\partial b_2}+\ldots\end{aligned}\right\};$$

and fourth, two of the three operators connected with the subsidiary equations (2) are

$$\left.\begin{aligned} D_7 = 0\,.\,a_0\frac{\partial}{\partial a_0}&+b_0\frac{\partial}{\partial b_0}+2c_0\frac{\partial}{\partial c_0}+3d_0\frac{\partial}{\partial d_0}+\ldots\\ &+(-1)\,a_1\frac{\partial}{\partial a_1}+0\,.\,b_1\frac{\partial}{\partial b_1}+1\,.\,c_1\frac{\partial}{\partial c_1}+\ldots\\ &+(-2)\,a_2\frac{\partial}{\partial a_2}+(-1)\,b_2\frac{\partial}{\partial b_2}+0\,.\,c_2\frac{\partial}{\partial c_2}+\ldots+(-3)\,a_3\frac{\partial}{\partial a_3}+\ldots,\\ D_9 = na_0\frac{\partial}{\partial a_0}&+(n-2)\,b_0\frac{\partial}{\partial b_0}+(n-4)\,c_0\frac{\partial}{\partial c_0}+\ldots\\ &+(n-1)\,a_1\frac{\partial}{\partial a_1}+(n-3)\,b_1\frac{\partial}{\partial b_1}+(n-5)\,c_1\frac{\partial}{\partial c_1}+\ldots\\ &\qquad+(n-2)\,a_2\frac{\partial}{\partial a_2}+(n-4)\,b_2\frac{\partial}{\partial b_2}+\ldots\\ &\qquad\qquad+(n-3)\,a_3\frac{\partial}{\partial a_3}+\ldots\end{aligned}\right\}$$

17. In obtaining the leading coefficients, which are the simultaneous solutions of the equations $D_1 = 0$ and $D_6 = 0$, the special form of those two equations leads to an easy method of classifying the solutions.

They may be regarded in two ways: first, they are the differential equations of the *simultaneous invariants of the system of binary quantics*

$$\begin{array}{l} (b_0,\ a_1 \between X,\ Y), \\ (c_0,\ b_1,\ a_2 \between X,\ Y)^2, \\ (d_0,\ c_1,\ b_2,\ a_3 \between X,\ Y)^3, \\ (e_0,\ d_1,\ c_2,\ b_3,\ a_4 \between X,\ Y)^4; \\ \dots\dots\dots\dots\dots \end{array}$$

second (which is practically another form of considering the first, and which is the way in which they will be regarded in subsequent applications), they are the differential equations of the *invariants and covariants of a system of simultaneous binary quantics, the coefficients of which are*

$$\begin{array}{l} c_0,\ b_1,\ a_2; \\ d_0,\ c_1,\ b_2,\ a_3; \\ e_0,\ d_1,\ c_2,\ b_3,\ a_4; \\ \dots\dots\dots\dots \end{array}$$

and the variables of which (and of their covariants) are a_1 *and* $-b_0$.

This interpretation is limited to unipartite quantics; a similar interpretation is found subsequently for bipartite quantics. It is to be noticed that, in the present case, the coefficient a_0 of the leading term of the fundamental quantic does not occur in either D_1 or D_6; and it is therefore a simultaneous solution which must, for the aggregate, be associated with the system of simultaneous binariants. With this addition, and regarding a_0 as a binary quantic of order zero, the former of the interpretations of the characteristic equations now is: they are the differential equations of the *simultaneous invariants of the system of binary quantics*

$$\begin{array}{l} (a_0 \between X,\ Y)^0, \\ (b_0,\ a_1 \between X,\ Y)^1, \\ (c_0,\ b_1,\ a_2 \between X,\ Y)^2, \\ (d_0,\ c_1,\ b_2,\ a_3 \between X,\ Y)^3, \\ (e_0,\ d_1,\ c_2,\ b_3,\ a_4 \between X,\ Y)^4, \\ \dots\dots\dots\dots\dots \end{array}$$

18. In the successive applications we shall limit the investigations to the deductions of concomitants which are algebraically independent of one another, and it is therefore necessary to indicate the number of these concomitants which must be expected in any case.

Let N be the total number of literal coefficients which exist in the most general forms of the quantic or of the system of simultaneous quantics, the algebraically independent concomitants of which are desired. Then, when the linear transformations are effected on the quantics, there are N equations connecting the new coefficients with the old; and all these equations involve the elements of the transformation.

In addition to these equations, we have three connecting the variables x and X, and three connecting the variables U and u; also one more given by

$$\Delta = \text{determinant of transformation.}$$

There are thus in all $N + 3 + 3 + 1 = N + 7$ equations which involve the elements of the transformation.

These $N + 7$ are not, however, independent; their number must be reduced by unity on account of the relation

$$u_x = U_X,$$

which, independent of the elements, is satisfied for all linear transformations. The number of equations, independent of one another and involving the elements of the transformation, is therefore

$$N + 6.$$

The number of elements of the transformation is 9.

When, therefore, we proceed to eliminate, among the independent equations, the elements of the transformation, we shall in the end have

$$(N + 6 - 9 =)\, N - 3$$

relations independent of one another, these relations involving x, X; u, U; Δ, and the coefficients. But we otherwise know that these relations are of the form

$$\phi(A, X, U) = \Delta^\lambda \phi(a, x, u),$$

and every such relation determines a concomitant. Hence *the number of concomitants algebraically independent of one another is* $N - 3$; *and in terms of these every concomitant can be algebraically expressed.*

It is to be understood that, in addition to these, we have the universal concomitant u_x.

A method of obtaining this number is indicated in §35 after some simple cases have been discussed.

PART II.

APPLICATIONS TO UNIPARTITE QUANTICS.

I.—*The Quadratic.*

19. In this case there are six coefficients, and the equation $D_1\psi = 0$ is

$$a_1 \frac{\partial\psi}{\partial b_0} + a_2 \frac{\partial\psi}{\partial b_1} + 2b_1 \frac{\partial\psi}{\partial c_0} = 0.$$

Forming the auxiliary equations necessary to obtain the most general solution of this equation, we have

$$\frac{da_0}{0} = \frac{da_1}{0} = \frac{da_2}{0} = \frac{db_0}{a_1} = \frac{db_1}{a_2} = \frac{dc_0}{2b_1}.$$

Of these five auxiliary equations it is necessary to have five independent integrals, which may evidently be taken in the form

$$\begin{aligned} \theta_0 &= a_0, \\ \theta_1 &= a_1, \\ \theta_2 &= a_2, \\ \theta_3 &= a_2 b_0 - a_1 b_1, \\ \theta_4 &= a_2 c_0 - b_1^2. \end{aligned}$$

Every solution ψ of the equation $D_1\psi = 0$ can, by the theory of this class of differential equations, be expressed as a functional combination of θ_0, θ_1, θ_2, θ_3, θ_4; and therefore, to obtain solutions common to $D_1\psi = 0$, $D_6\psi = 0$, we must take such functional combinations of θ_0, θ_1, θ_2, θ_3, θ_4 as satisfy $D_6\psi = 0$. Now

$$\begin{aligned} D_6\theta_0 &= 0, \\ D_6\theta_4 &= 0, \end{aligned}$$

so that θ_0 and θ_4 are common solutions of the two equations. And

$$\begin{aligned} D_6\theta_1 &= b_0, \\ D_6\theta_2 &= 2b_1, \\ D_6\theta_3 &= b_1 b_0 - a_1 c_0, \end{aligned}$$

so that
$$\theta_2 D_6 \theta_1 - \tfrac{1}{2}\theta_1 D_6 \theta_2 = \theta_3,$$
$$\theta_2 D_6 \theta_3 - \tfrac{1}{2}\theta_3 D_6 \theta_2 = -\theta_1\theta_4.$$

Hence, writing p for $\theta_1\theta_2^{-\frac{1}{2}}$ and q for $\theta_3\theta_2^{-\frac{1}{2}}$, we have
$$\theta_2 D_6 p = q, \quad \theta_2 D_6 q = -p\theta_4;$$
and therefore, since $D_6\theta_4 = 0$, we find
$$D_6(q^2 + p^2\theta_4) = 0,$$
so that the only functional combination other than θ_0 and θ_4 is
$$\begin{aligned}\phi &= q^2 + p^2\theta_4 \\ &= (\theta_3^2 + \theta_1^2\theta_4)\,\theta_2^{-1} \\ &= a_2 b_0^2 - 2a_1 b_1 b_0 + a_1^2 c_0,\end{aligned}$$
after substitution and reduction. And it follows that every solution common to $D_1\psi = 0$, $D_6\psi = 0$ is expressible in terms of θ_0, θ_4 and ϕ.

The subsidiary operators D_7 and D_9 are, for the present case,
$$D_7 = b_0\frac{\partial}{\partial b_0} + 2c_0\frac{\partial}{\partial c_0} - a_1\frac{\partial}{\partial a_1} - 2a_2\frac{\partial}{\partial a_2},$$
$$D_9 = 2a_0\frac{\partial}{\partial a_0} - 2c_0\frac{\partial}{\partial c_0} + a_1\frac{\partial}{\partial a_1} - b_1\frac{\partial}{\partial b_1}.$$

And by actual substitution we find
$$D_7\theta_0 = 0, \quad D_7\theta_4 = 0, \quad D_7\phi = 0;$$
$$D_9\theta_0 = 2\theta_0, \quad D_9\theta_4 = -2\theta_4, \quad D_9\phi = 0;$$
so that the subsidiary equations are satisfied provided the values of $m - p$ associated with θ_0, θ_4, and ϕ respectively are 2, -2, 0. That these are the values may be verified at once by the method of §14.

20. Considering now θ_0, we have $m - p = 2$, so that we determine p. But
$$D_2\theta_0 = \left(a_0\frac{\partial}{\partial a_1} + b_0\frac{\partial}{\partial b_1} + 2a_1\frac{\partial}{\partial a_2}\right)\theta_0 = 0;$$
by §7 it follows that $p = 0$ and therefore $m = 2$. Hence the concomitant is
$$U = a_0 x_1^2 + \ldots\ldots,$$
that is, it is the original quantic.

Considering now θ_4, we have $m-p=-2$, so that we determine m. But

$$D_3\theta_4=\left(c_0\frac{\partial}{\partial b_0}+2b_0\frac{\partial}{\partial a_0}+b_1\frac{\partial}{\partial a_1}\right)\theta_4=0;$$

so that by §8 it follows that $m=0$ and therefore $p=2$. Hence the concomitant is

$$\Theta=(a_2c_0-b_1^2)u_1^2+\ldots\,,$$

that is, it is the reciprocant of the original quantic.

Considering now ϕ, we have $m-p=0$, so that we determine either m or p. We have

$$D_2^2\phi=2(a_0^2c_0-a_0b_0^2),$$
$$D_3^3\phi=0,$$

so that by §7 it follows that $p=2$ and therefore $m=2$. Hence the concomitant is

$$\Phi=(a_2b_0^2-2a_1b_1b_0+a_1^2c_0)x_1^2u_1^2+\ldots.$$

Hence:

Every concomitant of the quadratic can be expressed in terms of U, Θ, and Φ.

21. A simple illustration of the principle of equivalent systems arises in the present case. We have

$$D_3\phi=-2a_0(b_1b_0-a_1c_0),\quad D_3\theta_0=0,$$
$$D_3\theta_4=\quad 2(a_1c_0-b_0b_1),$$

so that

$$D_3(\theta_0\theta_4-\phi)=0.$$

Hence $\theta_0\theta_4-\phi$, as the leading coefficient of a concomitant, has $p=0$; and evidently from the combination of θ_0 and θ_4 it has $m-p=0$, so that $m=0$; the function is an invariant. We therefore take

$$\begin{aligned}H&=\theta_0\theta_4-\phi\\&=a_0c_0a_2+2a_1b_1b_0-a_0b_1^2-a_2b_0^2-a_1^2c_0;\end{aligned}$$

and we evidently have

$$u_x^2H=U\Theta-\Phi,$$

so that we have an algebraically equivalent system given by U, Θ, H; and in terms of these three concomitants every concomitant can be expressed. This is the ordinary theory of the quadratic.

It will be noticed that the foregoing method of operators determines a non-resoluble concomitant from a given leading coefficient; the combination $\theta_0\theta_4-\phi$ determines H, and not u_x^2H.

II.—*The Cubic.*

22. In this case there are ten coefficients, and the characteristic equation $D_1\psi = 0$ is

$$a_1\frac{\partial\psi}{\partial b_0} + a_2\frac{\partial\psi}{\partial b_1} + a_3\frac{\partial\psi}{\partial b_2} + 2b_1\frac{\partial\psi}{\partial c_0} + 2b_2\frac{\partial\psi}{\partial c_1} + 3c_1\frac{\partial\psi}{\partial d_0} = 0.$$

Proceeding to obtain the general solution in the usual way, we form the auxiliary equations

$$\frac{da_0}{0} = \frac{da_1}{0} = \frac{da_2}{0} = \frac{da_3}{0} = \frac{db_0}{a_1} = \frac{dc_0}{2b_1} = \frac{dd_0}{3c_1} = \frac{db_1}{a_2} = \frac{dc_1}{2b_2} = \frac{db_2}{a_3},$$

nine in number; it is necessary that we should have nine independent integrals of these auxiliary equations in order to form the most general solution possible of $D_1\psi = 0$.

Now systems of nine independent integrals can be formed in several ways, and these systems are equivalent to one another; that is to say, they are such that the functions occurring in any one system can be uniquely expressed in terms of the functions occurring in any other system. Thus we may take

$$\begin{aligned}
&a_0,\ a_1,\ a_2,\ a_3,\\
&a_2b_0^2 - 2a_1b_1b_0 + a_1^2c_0,\\
&a_3b_0 - a_1b_2,\\
&a_2c_0 - b_1^2,\\
&a_3c_1 - b_2^2,\\
&a_3^2d_0 - 3a_3b_2c_1 + 2b_2^3,
\end{aligned}$$

which are independent of one another, and will therefore constitute a system of the kind required. And modifications can evidently be made among the members of a system, provided that the proper number of independent functions remain; thus, in virtue of the relation

$$a_2(a_2b_0^2 - 2a_1b_1b_0 + a_1^2c_0) = (a_2b_0 - a_1b_1)^2 + a_1^2(a_2c_0 - b_1^2),$$

we may replace a_2 by $a_2b_0 - a_1b_1$.

But it appears on trial (the work is not here reproduced) that the equations similar to those of §19, necessary for the deduction of those functional combinations of the integrals of the foregoing system which will satify $D_6\psi = 0$, are difficult to solve, though not at first sight difficult to form. The system of integrals, which seems to be the easiest to treat in this regard, is obtained by a

method similar to that which is adopted in a corresponding question in the theory of functional invariants* whereby one of the variables of the equation $D_1\psi = 0$ is made, so to speak, a "variable of reference"—a quantity in powers of which integrals are expressed. And the method has the additional advantage of an obvious purely mechanical generalization to quantics of any order.

23. We take, then, as the first of the system of nine integrals

$$\theta_0 = a_0$$

and have at once, replacing D_6 by Δ for convenience,

$$\Delta\theta_0 = 0,$$

so that a_0 is a solution common to the two characteristic equations.

We take as the second of the system

$$\theta_1 = a_1$$

and have

$$\Delta\theta_1 = b_0;$$

this quantity a_1 is taken as the "variable of reference."

We take as the third of the system

$$\theta_2 = a_2;$$

so that $\Delta\theta_2 = 2b_1$, and therefore

$$\theta_1\Delta\theta_2 - 2\theta_2\Delta\theta_1 = 2(a_1b_1 - a_2b_0).$$

Now it may be at once verified that

$$\theta_3 = a_1b_1 - a_2b_0$$

is a solution of the auxiliary equations, and so may be taken as the fourth of the system; and since

$$\Delta\theta_3 = a_1c_0 - b_1b_0,$$

we have

$$\theta_1\Delta\theta_3 - \theta_3\Delta\theta_1 = a_1^2c_0 - 2a_1b_1b_0 + a_2b_0^2.$$

It can similarly be verified that

$$\theta_4 = a_1^2c_0 - 2a_1b_1b_0 + a_2b_0^2$$

is a solution of the auxiliary equations, and it is therefore taken as the fifth of the system. And since

$$\Delta\theta_4 = 0,$$

it is a solution common to the two characteristic equations.

* "A class of functional invariants." Phil. Trans. (1889, A), pp. 71–118.

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Proceeding similarly, we take as the sixth of the system

$$\theta_5 = a_3,$$

so that, since $\Delta\theta_5 = 3b_2$, we have

$$\theta_1\Delta\theta_5 - 3\theta_5\Delta\theta_1 = 3(a_1b_2 - a_3b_0).$$

As before, it may be verified that

$$\theta_6 = a_1b_2 - a_3b_0$$

is a solution of the auxiliary equations, and may therefore be taken as the seventh of the system; and

$$\Delta\theta_6 = 2(a_1c_1 - b_0b_2),$$

so that $$\theta_1\Delta\theta_6 - 2\theta_6\Delta\theta_1 = 2(a_1^2c_1 - 2a_1b_0b_2 + a_3b_0^2).$$

Again, the quantity $$\theta_7 = a_1^2c_1 - 2a_1b_0b_2 + a_3b_0^2$$

is a solution of the auxiliary equations, and it may therefore be taken as the eighth of the system. We have

$$\Delta\theta_7 = a_1^2d_0 - 2a_1b_0c_1 + b_0^2b_2,$$

so that $$\theta_1\Delta\theta_7 - \theta_7\Delta\theta_1 = a_1^3d_0 - 3a_1^2b_0c_1 + 3a_1b_0^2b_2 - a_3b_0^3.$$

Lastly, the quantity

$$\theta_8 = a_1^3d_0 - 3a_1^2b_0c_1 + 3a_1b_0^2b_2 - a_3b_0^3$$

is a solution of the auxiliary equations, and it may therefore be taken as the ninth of the system. Moreover, since

$$\Delta\theta_8 = 0,$$

it is a solution common to the two equations $D_1\psi = 0$, $D_6\psi = 0$.

It follows from the forms of the nine quantities θ that they are independent of one another, for a_0 is introduced into the system by θ_0, a_1 by θ_1, a_2 by θ_2, b_1 by θ_3, c_0 by θ_4, a_3 by θ_5, b_2 by θ_6, c_1 by θ_7 and d_0 by θ_8 alone, so that among the quantities θ there can be no relation.

24. The process here adopted is one of general application. We take as the first integral of the auxiliary equations the coefficient of the highest power of x_1, and as the "variable of reference," the coefficient of the next lower power of x_1 which involves x_3; starting-points in the succession of integrals are given by coefficients of the terms in the quantic involving x_1 and x_3 only, and successive integrals are suggested by framing combinations of the type $\theta_1\Delta\theta_m - \lambda\theta_m\Delta\theta_1$. These combinations will be called Jacobian combinations.

25. Returning, now, to the equations given by the Jacobian combinations of the quantities θ with θ_1, and modifying them by the substitutions

$$\theta_0 = \phi_0;$$
$$\theta_2 = \theta_1^2\phi_2,\ \theta_3 = \theta_1\phi_3,\ \theta_4 = \phi_4;$$
$$\theta_5 = \theta_1^3\phi_5,\ \theta_6 = \theta_1^2\phi_6,\ \theta_7 = \theta_1\phi_7,\ \theta_8 = \phi_8;$$

we have them in the form

$$\begin{aligned}
\theta_1^2\Delta\phi_0 &= 0;\\
\theta_1^2\Delta\phi_2 &= 2\phi_3,\\
\theta_1^2\Delta\phi_3 &= \phi_4,\\
\theta_1^2\Delta\phi_4 &= 0;\\
\theta_1^2\Delta\phi_5 &= 3\phi_6,\\
\theta_1^2\Delta\phi_6 &= 2\phi_7,\\
\theta_1^2\Delta\phi_7 &= \phi_8,\\
\theta_1^2\Delta\phi_8 &= 0.
\end{aligned}$$

These are the equations which, when integrated, determine those functional combinations of the quantities θ which are to be solutions of the equation $D_6\psi = 0$. There must (§18) be obtained for this purpose seven independent integrals of these equations, and they may be taken in the forms

$$\begin{aligned}
\chi_1 &= \phi_0,\\
\chi_2 &= \phi_4,\\
\chi_3 &= \phi_2\phi_4 - \phi_3^2,\\
\chi_4 &= \phi_8,\\
\chi_5 &= \phi_6\phi_8 - \phi_7^2,\\
\chi_6 &= \phi_5\phi_8^2 - 3\phi_6\phi_7\phi_8 + 2\phi_7^3,\\
\chi_7 &= \phi_3\phi_8 - \phi_4\phi_7.
\end{aligned}$$

Every solution of the equation $D_6\psi = 0$, which is also a function of the quantities θ or the quantities ϕ and is therefore a solution of the equation $D_1\psi = 0$, can be expressed in terms of the quantities $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7$.

Hence *every solution common to the two equations* $D_1\psi = 0$, $D_6\psi = 0$, *can be expressed in terms of these seven solutions common to the two equations and algebraically independent of one another.*

26. The effects of the operators D_7 and D_9 on the quantities θ are as follows:

	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
D_7	0	$-\theta_1$	$-2\theta_2$	$-\theta_3$	0	$-3\theta_5$	$-2\theta_6$	$-\theta_7$	0
D_9	$3\theta_0$	$2\theta_2$	θ_2	$2\theta_3$	$3\theta_4$	0	θ_6	$2\theta_7$	$3\theta_8$

and therefore

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
D_7	0	0	0	0	0	0	0
D_9	$3\chi_1$	$3\chi_2$	0	$3\chi_4$	0	0	$3\chi_7$
Value of $m-p$	3	3	0	3	0	0	3

If, then, when the seven quantities χ are expressed in terms of the original coefficients of the quantic, the several values of $m-p$ for each of the quantities derived by the method of §14 should agree with the values as given in the last line of the table; then, by the preceding theory, each of the quantities χ is the leading coefficient, or the source, of a concomitant the full expression of which is obtainable by the methods previously given.

27. When actual substitution is made in the quantities χ and the resulting expressions are reduced, the following are their respective values:

$$\left.\begin{aligned}
v_0 &= \chi_1 = a_0, \\
v_2 &= \chi_2 = (c_0,\ b_1,\ a_2)(a_1,\ -b_0)^2, \\
h_2 &= \chi_3 = a_2c_0 - b_1^2, \\
v_3 &= \chi_4 = (d_0,\ c_1,\ b_2,\ a_3)(a_1,\ -b_0)^3, \\
h_3 &= \chi_5 = \left(\begin{array}{r|r|r} b_2d_0 & -b_2c_1 & a_3c_1 \\ -c_1^2 & +a_3d_0 & -b_2^2 \end{array}\right)(a_1,\ -b_0)^2, \\
\phi_3 &= \chi_6 = \left(\begin{array}{r|r|r|r} a_3d_0^2 & a_3c_1d_0 & -a_3b_2d_0 & -a_3^2d_0 \\ -3b_2c_1d_0 & -2b_2^2d_0 & +2a_3c_1^2 & +3a_3b_2c_1 \\ +2c_1^3 & +b_2c_1^2 & -b_2^2c_1 & -2b_2^3 \end{array}\right)(a_1,\ -b_0)^3, \\
j_{2,3} &= \chi_7 = \left(\begin{array}{r|r|r|r} b_1d_0 & a_2d_0 & -a_3c_0 & -a_3b_1 \\ -c_0c_1 & -2b_2c_0 & +2a_2c_1 & +a_2b_2 \\ & +b_1c_1 & -b_1b_2 & \end{array}\right)(a_1,\ -b_0)^3.
\end{aligned}\right\} \quad (13)$$

The determination of the quantities $m-p$ to be associated with these is made by the method of §14. Thus, in the case of χ_4, the weight of any term as $a_1^3 d_0$ is $W = 6\rho - 3$, while that of the term $c_1^3 a_0$, obtained from this by interchanging the coefficients of similar terms in x_1 and x_2, is $W' = 9\rho - 6$, so that for

$$m - p = \frac{W' - W}{\rho - 1} = 3,$$

agreeing with the value in the table. The values of $m-p$ thus derived for the quantities χ agree with the values in the table; and hence we infer that these quantities χ are leading coefficients of concomitants of the cubic. These concomitants will be denoted by U_0; U_2, H_2; U_3, H_3, Φ_3; $J_{2,3}$.

28. We now infer the general theorem:

Every concomitant of the ternary cubic can be expressed as a function of the concomitants U_0; U_2, H_2; U_3, H_3, Φ_3; $J_{2,3}$, and the universal concomitant u_x.

The universal concomitant u_x needs to be included; for any constant, say unity, is evidently a solution of the characteristic equations and yet the expression in terms of the seven concomitants is nugatory. The development of the concomitant is normal; for since $\psi_{0,0} = 1$ we have $m = p$ and $\phi_{0,0} = u_1$, so that $m = p = 1$. Also $\phi_{1,0} = u_2$; $\phi_{0,1} = u_3$, and thus we have $x_1u_1 + x_2u_2 + x_3u_3$ as the full expression.

It is a known theorem—here practically proved again—that u_x is the only irreducible universal concomitant for ternary quantics; it is therefore the only one that needs to be associated with the algebraically irreducible concomitants special to any quantic. And it is of importance in the expressions of algebraically reducible concomitants; for when a relation obtained through the leading coefficients is changed into one between the concomitants, it is necessary to associate with each term in the relation such a power of u_x as will render the order and the class uniform throughout.

29. Before these concomitants can be considered as fully given, it is yet necessary to determine for such of them one of the integers m or p and infer the other from the value of $m-p$.

First, for χ_1, we have $m - p = 3$, so that we determine p, using (9). Evidently $D_2\chi_1 = 0$, so that $p = 0$, and m is therefore 3. Thus

$$U_1 = \chi_1 x_1^3 + \dots \qquad (14),$$

being in fact the quantic itself.

Second, for χ_2, we have $m-p=3$, so that we determine p. We have

$$D_2\chi_2 = 2a_0(a_1c_0 - b_1b_0),$$
$$D_2^2\chi_2 = 2a_0(a_0c_0 - b_0^2),$$
$$D_2^3\chi_2 = 0,$$

so that $p=2$ and m is therefore 5. Thus

$$U_2 = \chi_2 x_1^5 u_1^2 + \ldots . \tag{15}$$

Third, for χ_3, we have $m-p=0$. To determine p we have

$$D_2\chi_3 = 2(a_1c_0 - b_1b_0),$$
$$D_2^2\chi_3 = 2(a_0c_0 - b_0^2),$$
$$D_2^3\chi_3 = 0,$$

so that $p=2=m$. Thus

$$H_2 = \chi_3 x_1^2 u_1^2 + \ldots . \tag{16}$$

In this connection one result may be noticed. From the forms of $D_2\chi_2$ and $D_2\chi_3$, we have

$$D_2(\chi_2 - \chi_1\chi_3) = 0.$$

The value of $m-p$ to be associated with $\chi_2 - \chi_1\chi_3$ is 3, and the equation just obtained shows that p is zero, so that $\chi_2 - \chi_1\chi_3$ is the leading coefficient of a pure covariant of the third order. It is the Hessian H, and we have the relation

$$u_x^2 H = U_2 - U_1 H_2.$$

Fourth, for χ_4, for which $m-p=3$, we determine p. We have

$$D_2^3\chi_4 = 6a_0(a_0^2d_0 - 3a_0b_0 + 2b_0^3),$$

so that $D_2^4\chi_4 = 0$; hence $p=3$ and m is therefore 6. Thus

$$U_3 = \chi_4 x_1^6 u_1^3 + \ldots . \tag{17}$$

Fifth, for χ_5, for which $m=p$. We find in a similar manner that $p=4$, so that $m=4$, and thus

$$H_3 = \chi_5 x_1^4 u_1^4 + \ldots . \tag{18}$$

Similarly, for χ_6, we find $p=6=m$, so that

$$\Phi_3 = \chi_6 x_1^6 u_1^6 + \ldots \tag{19},$$

and for χ_7 we find $p=4$ and $m=7$, so that

$$J_{2,3} = \chi_7 x_1^7 u_1^4 + \ldots . \tag{20}$$

It need hardly be remarked that in order to obtain the complete expressions of these seven fundamental concomitants, it is not necessary to effect all the differentiations implied by (I) and (II) for all the terms; for, since any concomitant is symmetric, either direct or skew, in variables and coefficients of terms of similar rank in the quantic, the knowledge of one term is sufficient to give by symmetric interchange all terms of similar rank in the concomitant.

30. The important connection of §17 between the theory of ternary quantics and that of simultaneous binary quantics, can here be made applicable immediately to the cubic. On inspection of the equations, it appears that they are the characteristic differential equations satisfied by covariants (and invariants) of the simultaneous binary quadratic and binary cubic having a_1 and $-b_0$ for variables; and that therefore *every simultaneous invariant or covariant of the binary quadratic and binary cubic having a_1 and $-b_0$ for variables, is the leading coefficient of a concomitant of the ternary cubic.* Moreover, this is practically the actual form in which the sources have been obtained, and it is on this account that the symbols v_2 (the binary quadratic), h_2 (the Hessian or discriminant of the quadratic), v_3 (the binary cubic), h_3 (the Hessian of the cubic), ϕ_3 (the cubicovariant of the cubic), and J_{23} (the Jacobian of the quadratic and the cubic), have been assigned to the sources.

Hence known results from the theory of the binary concomitants can be used for the theory of the ternary cubic; and conversely, the foregoing theory leads to an important inference as regards simultaneous binary quantics. The quantity a_0, thus interpreted, is in fact the binary quantic of order zero; that is, in the theory of simultaneous binary quantics it is a pure constant, and hence it follows that *every invariant and every covariant in the system which belongs to a binary cubic and a binary quadratic can be algebraically expressed in terms of the quadratic and its discriminant, the cubic with its Hessian and cubicovariant, and the Jacobian of the cubic and the quadratic.**

31. As illustrations of the general theorem that all the concomitants of the ternary quantic are expressible in terms of the given system, we proceed to some special cases.

* For the theory of the asyzygetic concomitants of the binary quadratic and binary cubic when simultaneous, see Salmon's *Higher Algebra*, §149, in 3d edition (1876); Clebsch's *Theorie der binären algebraischen Formen*, §59; Gordan's *Vorlesungen über Invariantentheorie*, §31 (a comparison of their notations is given in my memoir, "A class of functional invariants," Phil. Trans., 1889, A, p. 91, note), and Hammond, *The cubi-quadric system*, Amer. Journ. of Math., Vol. VIII (1886), pp. 138–155.

The cubic contravariant P has for its leading coefficient*

$$p = c_0(a_3c_1 - b_2^2) - b_1(a_3d_0 - b_2c_1) + a_2(b_2d_0 - c_1^2),$$

which is the invariant I of Salmon, the intermediate invariant of v_2 and h_3. Now if K be the Jacobian of two quadratics θ_1 and θ_2 of which the discriminants are D_1 and D_2 and intermediate invariant is I, then

$$K^2 = \theta_1\theta_2 I - \theta_1^2 D_2 - \theta_2^2 D_1.$$

Taking θ_2 to be v_2, we have $D_2 = h_2$; and taking θ_1 to be h_3, we have

$$\begin{aligned} 4D_1 &= 4(b_2d_0 - c_1^2)(a_3c_1 - b_2^2) - (a_3d_0 - b_2c_1)^2 \\ &= - \text{discriminant of cubic} \\ &= -\frac{1}{v_3^2}(\phi_3^2 + 4h_3^3). \end{aligned}$$

Also

$$Kv_3 = j_{23}h_3 + \frac{1}{2}v_2\phi_3,$$

for K, j_{23}, ϕ_3 are all Jacobians. Hence the relation becomes

$$\begin{aligned} pv_2h_3v_3^2 &= \left(j_{23}h_3 + \frac{1}{2}v_2\phi_3\right)^2 - \frac{1}{4}v_2^2(\phi_3^2 + 4h_3^3) + v_3^2h_3^2h_2 \\ &= j_{23}^2h_3^2 + j_{23}v_2\phi_3h_3 - v_2^2h_3^3 + v_3^2h_3^2h_2, \end{aligned}$$

so that

$$pv_2v_3^2 = j_{23}^2h_3 + j_{23}v_2\phi_3 - v_2^2h_3^2 + v_3^2h_2h_3.$$

Hence passing to the concomitants of which the foregoing quantities are leading coefficients, we have

$$Pu_xU_2U_3^2 = J_{23}^2H_3 + J_{23}U_2\Phi_3 - U_2^2H_3^2 + U_3^2H_2H_3,$$

the factor u_x being inserted to make the order and the class uniform.

The reciprocant F (Cayley, l. c., p. 644) has for its leading coefficient

$$\begin{aligned} f &= a_3^2d_0^2 - 6a_3b_2c_1d_0 + 4a_3c_1^3 + 4d_0b_2^3 - 3b_2^2c_1^2 \\ &= \frac{1}{v_3^2}(\phi_3^2 + 4h_3^3), \end{aligned}$$

and therefore

$$FU_3^2 = \Phi_3^2 + 4H_3^3,$$

no factor u_x being necessary.

The value of the quartinvariant S (Cayley, l. c., p. 641) is

$$S = v_0p - h_3 + h_2^2 + l_2$$

* Cayley, *Third Memoir on Quantics*, Phil. Trans., 1856, p. 642.

(in the table the quantity $-cf^2h$ should be $-cfh^2$), where

$$l_2 = a_1\{d_0a_2b_1 - c_1(a_2c_0 + 2b_1^2) + 3b_2c_0b_1 - a_3c_0^2\}$$
$$- b_0\{d_0a_2^2 - 3c_1a_2b_1 + b_2(a_2c_0 + 2b_1^2) - a_3c_0b_1\},$$

being in fact the linear covariant L_2 of Salmon ($= q$ of Gordan); and it has already (§30) been proved that l_2 must be expressible in terms of the seven functions (13). In fact it is not difficult to verify that

$$l_2v_2v_3^2 = -\phi_3v_2^3 + j_{23}(h_2v_3^3 + 3h_3v_2^2 + j_{23}^2).$$

Hence

$$Sv_2v_3^2 = v_0v_2v_3^2p - v_2v_3^3h_3 + v_2v_3^2h_2^2 - \phi_3v_2^3 + j_{23}h_2v_3^3 + 3h_3v_2^2j_{23} + j_{23}^3,$$

the relation among leading coefficients. This, when turned into the corresponding relation between the concomitants, is

$$SU_2U_3^2u_x^4 = U_1(J_{23}^2H_3 + J_{23}U_2\Phi_3 - U_2^2H_3^2 + U_3^2H_2H_3)$$
$$- U_2U_3^3H_3 + U_2U_3^2H_2^2 - \Phi_3U_2^3 + J_{23}H_2U_3^3 + 3H_3U_2^2J_{23} + J_{23}^3,$$

the power of u_x being inserted to make the order and the class uniform throughout the equation.

And in every case the first step in the expression of a given concomitant in terms of the fundamental concomitants is the arrangement of its leading coefficients in powers of a_1 and b_0 (i. e. of h and j in Cayley's tables).

32. As a last illustration we may take the following: It is a consequence of the general theory that each of the eight quantities U, H, Ψ; P, Q, F; S, T* is expressible in terms of the seven fundamental concomitants and of u_x. Hence some relation must subsist among these eight quantities and u_x, an irrational form of which relation can be obtained by the following indications.

Taking U in the form $x^3 + y^3 + z^3 + 6lxyz$, we have

$$x^3 + y^3 + z^3 = \lambda,$$
$$y^3z^3 + z^3x^3 + x^3y^3 = \mu,$$
$$xyz = r,$$

where λ, μ, r are expressible in terms of U, H, Ψ and the coefficient l. Now if $x + y + z = 3p$, $yz + zx + xy = 3q$, we have

$$p(p^2 - q) = \frac{1}{27}(\lambda - 3r) = \sigma,$$

$$q(q^2 - pr) = \frac{1}{27}(3r^2 - \mu) = \rho, \text{ say,}$$

* Cayley's 34 Concomitants of the Ternary Cubic, *Amer. Journ. of Math.*, Vol. IV (1881), p. 1.

where σ and ρ are similarly expressed to λ, μ, r. From these it follows that if $p^3 = Y$, we have

$$Y^3 - Y^2(3\sigma + r) + Y(3\sigma^2 + r\sigma - \rho) - \sigma^3 = 0. \tag{i}$$

Similarly, if

$$\begin{aligned} \xi^3 + \eta^3 + \zeta^3 &= \lambda', \\ \eta^3\zeta^3 + \zeta^3\xi^3 + \xi^3\eta^3 &= \mu', \\ \xi\eta\zeta &= r', \end{aligned}$$

where λ', μ', r' are expressible in terms of P, Q, F and the coefficient l, and if we write $\xi + \eta + \zeta = 3p' = 3\sqrt[3]{Y'}$, then

$$Y'^3 - Y'^2(3\sigma' + r') + Y'(3\sigma'^2 + r'\sigma' - \rho') - \sigma'^3 = 0, \tag{ii}$$

and both Y and Y' (and therefore p and p') can be expressed in irrational forms by means of (i) and (ii) in terms of the three covariants, the three contravariants and the coefficient l.

Now x, y, z being the roots of

$$u^3 - 3pu^2 + 3qu - r = 0,$$

and ξ, η, ζ those of

$$u^3 - 3p'u^2 + 3q'u - r' = 0,$$

then (Burnside and Panton, Theory of Equations, 1st edit. p. 113)

$$u_x = x\xi + y\eta + z\zeta = 3t$$

satisfies the equation

$$(pp' - t)^3 - 3hh'(pp' - t) + \tfrac{1}{2}(gg' \pm \sqrt{\Delta\Delta'}) = 0, \tag{iii}$$

where

$$\begin{aligned} h &= q - p^2 = -\frac{\sigma}{p}, \\ g &= -r + 3pq - 2p^3 = -r - 3\sigma + p^3, \end{aligned}$$

and

$$\Delta = g^2 + 4h^3,$$

with similar values for h', g', Δ'.

When the value of $p\,(=\sqrt[3]{Y})$ derived from (i) is substituted in h, g, Δ, they are expressed (irrationally) in terms of l and the three covariants; and when the value of $p'\,(=\sqrt[3]{Y'})$ derived from (ii) is substituted in h', g', Δ', they are similarly expressed in terms of l and the three contravariants. When both these sets of quantities and the values of p and p' are substituted in (iii), it comes to be an equation between U, H, Ψ; P, Q, F; u_x, and l. When its rationalized equivalent is obtained, it follows—from the fact that all the occurring quanti-

ties, other than combinations of l, are covariantive—that such combinations are also covariantive; that is to say, they can be expressed in terms of S and T.* This rational form would be the required relation.

Symbolical Representation of the Concomitants.

33. Instead of determining the order and the class by means of equations (9) and (11), the following method is effective, viz. *to change the leading coefficient into one which is symbolical in the umbral elements of the original quantic, and complete this symbolical expression according to the laws which apply to the concomitants of ternary quantics.*

For this purpose let

$$U_1 = a_0 x_1^3 + \ldots\ldots = \alpha_x^3 = \beta_x^3 = \gamma_x^3 = \ldots\ldots$$

Then $$h_2 = a_2 c_0 - b_1^2 = \tfrac{1}{2}\alpha_1\beta_1(\beta_2\alpha_3)^2,$$

and therefore $$H_2 = \tfrac{1}{2}(\alpha\beta u)^2\alpha_x\beta_x.$$

Next we have

$$v_2 = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0)^2 = \alpha_1\varepsilon_\xi^2$$

for $c_0 = \alpha_1\alpha_2^2$, $b_1 = \alpha_1\alpha_2\alpha_3$, $a_2 = \alpha_1\alpha_3^2$, and

$$\begin{aligned}\varepsilon_\xi &= \alpha_2 a_1 - \alpha_3 b_0 \\ &= \beta_1^2(\alpha_2\beta_3).\end{aligned}$$

Hence, remembering that with the symbolical notation, the repetition of a real coefficient requires the introduction of a new umbral coefficient, we have

$$v_2 = \alpha_1 . \beta_1^2(\alpha_2\beta_3) . \gamma_1^2(\alpha_3\gamma_3),$$

and therefore $$U_2 = (\alpha\beta u)(\alpha\gamma u)\,\alpha_x\beta_x^2\gamma_x^2.$$

Next we have $$v_3 = (d_0,\ c_1,\ b_2,\ a_3 \between a_1,\ -b_0)^3 = \theta_\xi^3,$$

where $$\theta_\xi = \alpha_2 a_1 - \alpha_3 b_0 = (\alpha_2\beta_3)\,\beta_1^2,$$

so that as before $$v_3 = (\alpha_2\beta_3)\,\beta_1^2 . (\alpha_2\gamma_3)\,\gamma_1^2 . (\alpha_3\delta_3)\,\delta_1^2,$$

and therefore $$U_3 = (\alpha\beta u)(\alpha\gamma u)(\alpha\delta u)\,\beta_x^2\gamma_x^2\delta_x^2.$$

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* It is not sufficient to have one only of the two invariants, for in a less special form we should have the quantities l^3 and abc, in the notation of Cayley's paper on the 34 concomitants.

Further, h_3 is the Hessian of v_3, so that we may write

$$h_3 = \frac{1}{2}(\theta\phi)^2)\,\theta_\xi\phi_\xi$$
$$= \frac{1}{2}(\alpha_2\beta_3)^2.(\alpha_2\gamma_3)\gamma_1^2.(\beta_2\delta_3)\,\delta_1^2,$$

and therefore
$$H_3 = \frac{1}{2}(\alpha\beta u)^2(\alpha\gamma u)(\beta\delta u)\,\gamma_x^2\delta_x^2.$$

Again, ϕ_3 is the cubicovariant of v_3, so that

$$\phi_3 = (\theta\phi)^2(\theta\psi)\,\phi_\xi\psi_\xi^2$$
$$= (\alpha_2\beta_3)^2(\alpha_2\gamma_3).(\beta_2\delta_3)\,\delta_1^2\,[(\gamma_2\varepsilon_3)\,\varepsilon_1^2.(\gamma_2\lambda_3)\,\lambda_1^2],$$

and therefore
$$\frac{\Phi}{3} = (\alpha\beta u)^2(\alpha\gamma u)(\beta\delta u)(\gamma\varepsilon u)(\gamma\lambda u)\,\delta_x^2\varepsilon_x^2\lambda_x^2.$$

And lastly, j_{23} is the Jacobian of v_2 and v_3, so that

$$j_{23} = \alpha_1(\varepsilon\theta)\,\varepsilon_\xi\theta_\xi^2.$$

We write $\varepsilon_1 = \alpha_2$ and $\varepsilon_2 = \alpha_3$, $\theta_1 = \beta_2$ and $\theta_2 = \beta_3$, and so on, so that

$$j_{23} = \alpha_1(\alpha_2\beta_3).(\alpha_2\gamma_3)\gamma_1^2\,[(\beta_2\delta_3)\,\delta_x^2.(\beta_2\varepsilon_3)\,\varepsilon_x^2],$$

and therefore
$$J_{23} = (\alpha\beta u)(\alpha\gamma u)(\beta\delta u)(\beta\varepsilon u)\,\alpha_x\gamma_x^2\delta_x^2\varepsilon_x^2.$$

It will be noticed that for each of the functions thus represented, the order and the class agree with their former values.

Modification of the Fundamental System.

34. According to a well known proposition, the square of any Jacobian binariant can be expressed in terms of other concomitants associated with the binary quantics of which the Jacobian is taken, and as the Jacobian may thus be considered to be of ambiguous sign, it may be deemed desirable to replace the two Jacobians in the foregoing system, viz. ϕ_3 and j_{23}, by equivalent concomitants of determinate sign.

For the first of them we have

$$\phi_3^2 = fv_3^2 - 4h_3^3,$$

where f is the discriminant of v_3, already (§31) considered, and thus replace ϕ_3 by f, which written symbolically is

$$(\alpha_2\beta_3)^2(\gamma_2\delta_3)^2(\alpha_2\gamma_3)(\beta_2\delta_3),$$

we have the corresponding function given by

$$F=(\alpha\beta u)^2(\gamma\delta u)^2(\alpha\gamma u)(\beta\delta u).$$

This replaces Φ_8.

To replace j_{23} we take the quantity p (of §31), the symbolical form of which is

$$p=\alpha_1(\epsilon\eta)^3,$$

where

$$\eta_x^2=h_8=\frac{1}{2}(\theta\phi)^2\theta_x\phi_x,$$

so that

$$p=\frac{1}{2}\alpha_1(\theta\phi)^2(\epsilon\theta)(\epsilon\phi)$$

$$=\frac{1}{2}\alpha_1(\beta_2\gamma_3)^2(\alpha_2\gamma_3)(\alpha_2\beta_3).$$

This leading coefficient determines a function

$$\frac{1}{2}(\beta\gamma u)^2(\alpha\gamma u)(\alpha\beta u)\alpha_x$$

$$=\frac{1}{6}(\alpha\beta u)(\alpha\gamma u)(\beta\gamma u)(\alpha\beta\gamma)u_x,$$

by the usual method of compounding these determinants; hence there is effectively determined a function

$$P=\frac{1}{6}(\alpha\beta\gamma)(\alpha\beta u)(\alpha\gamma u)(\beta\gamma u).$$

This replaces J_{23}; and the present rejection of the factor u_x accounts for its insertion in §31, necessary to render the order and the class both uniform throughout the equation which gives the expression for P in terms of the former fundamental system.

The Number of Algebraically Independent Concomitants of the Ternary n^{tic}.

35. Before proceeding to the detailed consideration of the quartic, the general method of obtaining the proper (§18) number of the independent solutions of the equations $D_1\psi=0$, $D_6\psi=0$ can now be indicated.

We find as before a complete set of independent solutions $\theta_0, \theta_1, \theta_2, \ldots$ of $D_1\psi=0$, and then take such functional combinations of them, say $f(\theta_0, \theta_1, \ldots)$, as will satisfy $D_6\psi=0$ or $\Delta\psi=0$; we must therefore have

$$0=\frac{\partial f}{\partial\theta_0}\Delta\theta_0+\frac{\partial f}{\partial\theta_1}\Delta\theta_1+\frac{\partial f}{\partial\theta_2}\Delta\theta_2+\ldots.$$

Hence the subsidiary equations necessary for the determination of f are

$$\frac{d\theta_0}{\Delta\theta_0}=\frac{d\theta_1}{\Delta\theta_1}=\frac{d\theta_2}{\Delta\theta_2}=\ldots\ldots;$$

the number of independent solutions of this set will give the required number of independent functional combinations. Now all the quantities $\Delta\theta$ are not functions of the variables θ of these equations; it is necessary to take such combinations of the $\Delta\theta$ as are expressible in terms of the variables. In actual practice these combinations are similar to those which arose in the case of the cubic (§25), viz. functions of quotients of the variables θ such that when a quotient is operated on by Δ, the result is expressible in terms of some other quotient.

To estimate the effect of these modifications, let us consider them in connection with the ternary quantic of order n, which has $\frac{1}{2}(n+1)(n+2)$ coefficients. The number of subsidiary equations associated with $D_1\psi=0$ is less than this integer by unity, and therefore the number of the quantities θ being the number of independent integrals of these equations, is

$$\tfrac{1}{2}(n^2+3n).$$

In forming the functional combinations of the quantities $\Delta\theta$, it is necessary (§23) to take some one of the quantities θ, as θ_1, for a variable of reference, and then the number of independent equations of the form

$$\theta_1\Delta\theta_r-\lambda\theta_r\Delta\theta_1=\theta_s,$$

which can be formed, is $\frac{1}{2}(n^2+3n)-1$. Each such equation can be used for the transformation of a fraction in the subsidiary equations

$$\frac{d\theta_0}{\Delta\theta_0}=\frac{d\theta_1}{\Delta\theta_1}=\ldots\ldots,$$

and therefore the number of equations in the modified set being one less than the number of modified fractions, is $\frac{1}{2}(n^2+3n)-2$. But each of these modified subsidiary equations leads to an integral, and therefore the number of independent integrals is $\frac{1}{2}(n^2+3n)-2$, which is the number $\frac{1}{2}(n+1)(n+2)-3$ of §18,

$$=\tfrac{1}{2}(n+4)(n-1),$$

which is the required number of algebraically independent solutions of the simultaneous partial differential equations $D_1\psi = 0$, $D_6\psi = 0$.

Since each such solution determines a concomitant, we have the result:

All the concomitants of the uni-ternary n^{ic} can be algebraically expressed in terms of u_x and of $\frac{1}{2}(n+4)(n-1)$ properly chosen independent concomitants.

Thus there are 3 for the quadratic in this algebraically complete system; there are 7 for the cubic, as was proved, and, as we shall now see, there are 12 for the quartic.

In the same way it may be proved that:

All the concomitants of the bi-ternary $n^o m^{ic}$, symbolically represented by $a_x^n u_\alpha^m$, can be algebraically expressed in terms of $\frac{1}{4}(n+1)(n+2)(m+1)(m+2)-3$ properly chosen independent concomitants.

III.—*The Quartic.*

36. The explicit form of the general quartic is

$$\begin{aligned}
&a_0x_1^4 + 4x_3a_1x_1^3 + 6x_3^2a_2x_1^2 + 4x_3^3a_3x_1 + a_4x_3^4,\\
&+ 4b_0x_1^3x_2 + 4x_3 3b_1x_1^2x_2 + 6x_3^2 2b_2x_1x_2 + 4x_3^3b_3x_2\\
&+ 6c_0x_1^2x_2^2 + 4x_3 3c_1x_1x_2^2 + 6x_3^2c_2x_2^2\\
&+ 4d_0x_1x_2^3 + 4x_3d_1x_2^3\\
&+ e_0x_2^4,
\end{aligned}$$

and the characteristic equations $D_1\psi = 0$, $D_6\psi = 0$ are respectively

$$\begin{aligned}
D_1 = a_1\frac{\partial}{\partial b_0} + 2b_1\frac{\partial}{\partial c_0} + 3c_1\frac{\partial}{\partial d_0} + 4d_1\frac{\partial}{\partial c_0} + a_2\frac{\partial}{\partial b_1} + 2b_2\frac{\partial}{\partial c_1}&\\
+ 3c_2\frac{\partial}{\partial d_1} + a_3\frac{\partial}{\partial b_2} + 2b_3\frac{\partial}{\partial c_2} + a_4\frac{\partial}{\partial b_3} &= 0,\\
D_6 = \Delta = b_0\frac{\partial}{\partial a_1} + 2b_1\frac{\partial}{\partial a_2} + 3b_2\frac{\partial}{\partial a_3} + 4b_3\frac{\partial}{\partial a_4} + c_0\frac{\partial}{\partial b_1} + 2c_1\frac{\partial}{\partial b_2}&\\
+ 3c_2\frac{\partial}{\partial b_3} + d_0\frac{\partial}{\partial c_1} + 2d_1\frac{\partial}{\partial c_2} + e_0\frac{\partial}{\partial d_1} &= 0.
\end{aligned}$$

To find the common solutions, it is first necessary to construct the subsidiary equations for the former of these, being 14 in number since there are 15 coefficients, and so 14 independent integrals of them are necessary.

The form of nine of these subsidiary equations is exactly the same as for the cubic, so that their integrals are the same as for the cubic, viz. we take

$$\theta_0,\ \theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6,\ \theta_7,\ \theta_8,$$

as in §23. For the remainder, which are

$$\frac{da_0}{0} = \ldots = \frac{da_4}{0} = \frac{db_3}{a_4} = \frac{dc_2}{2b_3} = \frac{dd_1}{3c_2} = \frac{de_0}{4d_1},$$

we proceed as before, and take

$$\theta_9 = a_4.$$

Then $\Delta\theta_9 = 4b_3$, so that

$$\theta_1\Delta\theta_9 - 4\theta_9\Delta\theta_1 = 4\theta_{10},$$

where

$$\theta_{10} = b_3a_1 - a_4b_0$$

is the next integral. We now have

$$\theta_1\Delta\theta_{10} - 3\theta_{10}\Delta\theta_1 = 3\theta_{11},$$

where

$$\theta_{11} = c_2a_1^2 - 2b_3a_1b_0 + a_4b_0^2$$

is another integral. Next

$$\theta_1\Delta\theta_{11} - 2\theta_{11}\Delta\theta_1 = 2\theta_{12},$$

where

$$\theta_{12} = d_1a_1^3 - 3c_2a_1^2b_0 + 3b_3a_1b_0^2 - a_4b_0^3$$

is the succeeding integral; and

$$\theta_1\Delta\theta_{12} - \theta_{12}\Delta\theta_1 = \theta_{13},$$

where

$$\theta_{13} = e_0a_1^4 - 4d_1a_1^3b_0 + 6c_2a_1^2b_0^2 - 4b_3a_1b_0^3 + a_4b_0^4$$

is the last integral; moreover, we have

$$\Delta\theta_{13} = 0.$$

37. By the substitutions

$$\theta_9\theta_1^{-4} = \phi_9,\quad \theta_{10}\theta_1^{-3} = \phi_{10},\quad \theta_{11}\theta_1^{-2} = \phi_{11},\quad \theta_{12}\theta_1^{-1} = \phi_{12},\quad \theta_{13}\theta_1^{-0} = \phi_{13},$$

these equations take the same forms as the eight equations of §25, viz.

$$\begin{aligned} \theta_1^2\Delta\phi_9 &= 4\phi_{10}, \\ \theta_1^2\Delta\phi_{10} &= 3\phi_{11}, \\ \theta_1^2\Delta\phi_{11} &= 2\phi_{12}, \\ \theta_1^2\Delta\phi_{12} &= \phi_{13}, \\ \theta_1^2\Delta\phi_{13} &= 0\ , \end{aligned}$$

so that of the $(8+5=)\,13$ equations we must have 12 integrals. Seven of these are already obtained, being $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7$; the remaining five are easily found to be

$$
\begin{aligned}
\chi_8 &= \phi_{13}, \\
\chi_9 &= \phi_{11}\phi_{13} - \phi_{12}^2, \\
\chi_{10} &= \phi_{10}\phi_{13}^2 - 3\phi_{11}\phi_{12}\phi_{13} + 2\phi_{12}^3, \\
\chi_{11} &= \phi_9\phi_{13} - 4\phi_{10}\phi_{12} + 3\phi_{11}^2, \\
\chi_{12} &= \phi_8\phi_{13} - \phi_4\phi_{12}.
\end{aligned}
$$

The twelve integrals are independent of one another, and every solution common to the two equations $D_1\psi = 0 = D_6\psi$ can be algebraically expressed in terms of $\chi_1, \chi_2, \ldots, \chi_{11}, \chi_{12}$.

38. The effects of the operators D_7 and D_9 on the quantities θ in the case of the quartic are as follows:

	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}
D_7	0	$-\theta_1$	$-2\theta_2$	$-\theta_3$	0	$-3\theta_5$	$-2\theta_6$	$-\theta_7$	0	$-4\theta_9$	$-3\theta_{10}$	$-2\theta_{11}$	$-\theta_{12}$	0
D_9	$4\theta_0$	$3\theta_1$	$2\theta_2$	$4\theta_3$	$6\theta_4$	θ_5	$3\theta_6$	$5\theta_7$	$7\theta_8$	0	$2\theta_{10}$	$4\theta_{11}$	$6\theta_{12}$	$8\theta_{13}$

and therefore

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
D_7	0	0	0	0	0	0	0	0	0	0	0	0
D_9	$4\chi_1$	$6\chi_2$	$2\chi_3$	$7\chi_4$	$4\chi_5$	$6\chi_6$	$8\chi_7$	$8\chi_8$	$6\chi_9$	$9\chi_{10}$	$-4\chi_{11}$	$9\chi_{12}$
Value of $m-p$	4	6	2	7	4	6	8	8	6	9	-4	9

Hence, if ultimately it should appear that the values of $m-p$ to be associated with the respective sources are as given in the last line of the second table, the twelve quantities χ are sources of concomitants.

39. When actual substitution is made in the quantities χ and the resulting expressions are reduced, the values of $\chi_1, \ldots, \chi_7$ are as given in (13), but it must be remembered that the coefficients are now coefficients of the quartic, and therefore different values of m and of p may need to be associated with those quantities. For the remainder we have

$$v_4 = \chi_8 = (e_0,\ d_1,\ c_2,\ b_3,\ a_4 \between a_1,\ -b_0)^4,$$

$$h_4 = \chi_9 = \left|\begin{array}{c|c|c|c|c} +e_0c_2 & +2e_0b_3 & +e_0a_4 & +2d_1a_4 & +a_4c_2 \\ -d_1^2 & -2c_2d_1 & +2d_1b_3 & -2b_3c_2 & -b_3^2 \\ & & -3c_2^2 & & \end{array}\right| \between a_1,\ -b_0)^4$$

$$\phi_4 = \chi_{10} = \left|\begin{array}{c|c|c|c|c|c|c} +e_0^2b_3 & +e_0^2a_4 & +5e_0d_1a_4 & -10e_0b_3^2 & -5e_0b_3a_4 & -e_0a_4^2 & -d_1a_4^2 \\ -3e_0d_1c_2 & +2e_0b_3d_1 & -15e_0c_2b_3 & +10d_1^2a_4 & +15d_1c_2a_4 & -2d_1b_3a_4 & +3c_2b_3a_4 \\ +2d_1^3 & -9e_0c_2^2 & +10d_1^2b_3 & & -10d_1b_3^2 & +9c_2^2a_4 & -2b_3^3 \\ & +6d_1^2c_2 & & & & -6c_2b_3^2 & \end{array}\right| \between a_1,\ -b_0)^6$$

$$i_4 = \chi_{11} = e_0a_4 - 4d_1b_3 + 3c_2^2,$$

$$j_{24} = \chi_{12} = \left|\begin{array}{c|c|c|c|c} +e_0b_1 & +e_0a_2 & +3d_1a_2 & -a_4c_0 & +b_3a_2 \\ -c_0d_1 & -3c_2c_0 & -3b_3c_0 & +3c_2a_2 & -a_4b_1 \\ & +2b_1d_1 & & -2b_3b_1 & \end{array}\right| \between a_1,\ -b_0)^4 \qquad (21)$$

The reasons of the notations adopted are fairly obvious; h_4 is the Hessian, ϕ_4 the cubicovariant, i_4 the quadrinvariant, of v_4 regarded as a quartic; and j_{24} is the Jacobian of v_2, regarded as a quadratic, and v_4.

40. To find the values of m and p, the shortest method will be to change all the expressions into symbolical forms. For this purpose, let

$$U_1 = a_0x_1^4 + \ldots = a_x^4 = \beta_x^4 = \gamma_x^4 = \ldots\ldots$$

be the original quartic. It is evident that U_1 is the concomitant with the source v_0.

Then
$$h_2 = a_2c_0 - b_1^2 = \tfrac{1}{2}\, a_1^2\beta_1^2\, (\beta_2 a_3)^2,$$

and therefore
$$\begin{aligned} H_2 &= \tfrac{1}{2}\,(\alpha\beta u)^2\,\alpha_x^2\beta_x^2 \\ &= \chi_8 x_1^4 u_1^2 + \ldots \end{aligned} \qquad (22).$$

Next we have $v_2 = (c_0, b_1, a_2 \between a_1, -b_0)^2 = \alpha_1^2 \epsilon_\xi^2,$

for $c_0 = \alpha_1^2\alpha_2^2$, $b_1 = \alpha_1^2\alpha_2\alpha_3$, $a_2 = \alpha_1^2\alpha_3^2$; and

$$\begin{aligned}\epsilon_\xi &= a_2a_1 - a_3b_0\\ &= \beta_1^3(\alpha_2\beta_3),\end{aligned}$$

so that $v_2 = \alpha_1^2 . \beta_1^3(\alpha_2\beta_3) . \gamma_1^3(\alpha_2\gamma_3),$

and therefore

$$\begin{aligned}U_2 &= (\alpha\beta u)(\alpha\gamma u)\,\alpha_x^2\beta_x^3\gamma_x^3\\ &= \chi_3 x_1^8 u_1^2 + \dots . \qquad (23).\end{aligned}$$

(As in the case of the cubic, we have $U_2 - U_1H_2$ divisible by u_x^2 and leaving as its other factor the Hessian of the quartic.)

Next we have

$$v_3 = (d_0, c_1, b_2, a_3 \between a_1, -b_0)^3 = \alpha_1\theta_\xi^3,$$

where $\theta_\xi = a_2a_1 - a_3b_0 = \beta_1^3(\alpha_2\beta_3),$

so that $v_3 = \alpha_1 . \beta_1^3(\alpha_2\beta_3) . \gamma_1^3(\alpha_2\gamma_3) . \delta_1^3(\alpha_2\delta_3);$

and therefore

$$\begin{aligned}U_3 &= (\alpha\beta u)(\alpha\gamma u)(\alpha\delta u)\,\alpha_x\beta_x^3\gamma_x^3\delta_x^3\\ &= \chi_4 x_1^{10}u_1^3 + \dots . \qquad (24).\end{aligned}$$

Next, h_3 is the Hessian of v_3, so that

$$\begin{aligned}h_3 &= \tfrac{1}{2}\,\alpha_1\beta_1(\theta\phi)^2\theta_\xi\phi_\xi\\ &= \tfrac{1}{2}\,\alpha_1\beta_1(\alpha_2\beta_3)^2 . (\alpha_2\gamma_3)\,\gamma_1^3 . (\beta_2\delta_3)\,\delta_1^3,\end{aligned}$$

and therefore

$$\begin{aligned}H_3 &= \tfrac{1}{2}\,(\alpha\beta u)^2(\alpha\gamma u)(\beta\delta u)\,\alpha_x\beta_x\gamma_x^3\delta_x^3\\ &= \chi_5 x_1^8 u_1^4 + \dots . \qquad (25).\end{aligned}$$

Again, ϕ_3 is the cubicovariant of v_3, so that

$$\begin{aligned}\phi_3 &= \alpha_1\beta_1\gamma_1(\theta\phi)^2\phi_\xi\psi_\xi^2(\theta\psi)\\ &= \alpha_1\beta_1\gamma_1(\alpha_2\beta_3)^2(\alpha_2\gamma_3) . \delta_1^3(\beta_2\delta_3) . (\gamma_2\lambda_3)\,\lambda_1^3(\gamma_2\mu_3)\,\mu_1^3,\end{aligned}$$

and therefore

$$\begin{aligned}\Phi_3 &= (\alpha\beta u)^2(\alpha\gamma u)(\beta\delta u)(\gamma\lambda u)(\gamma\mu u)\,\alpha_x\beta_x\gamma_x\delta_x^3\lambda_x^3\mu_x^3\\ &= \chi_6 x_1^{12}u_1^6 + \dots . \qquad (26).\end{aligned}$$

Also, j_{23} is the Jacobian of v_2 and v_3, so that

$$\begin{aligned}j_{23} &= \alpha_1^2\beta_1(\epsilon\theta)\,\epsilon_\xi\theta_\xi^2\\ &= \alpha_1^2\beta_1(\alpha_2\beta_3) . (\alpha_2\gamma_3)\,\gamma_1^3\,[(\beta_2\delta_3)\,\delta_1^3 . (\beta_2\lambda_3)\,\lambda_1^3],\end{aligned}$$

and therefore

$$J_{23} = (\alpha\beta u)(\alpha\gamma u)(\beta\delta u)(\beta\lambda u)\, \alpha_x^2\beta_x\gamma_x^3\delta_x^3\lambda_x^3$$
$$= \chi_7 x_1^{13} u_1^4 + \ldots . \qquad (27).$$

Coming now to the new forms in (21), we have

$$v_4 = (e_0,\, d_1,\, c_2,\, b_3,\, a_4)(a_1,\, -b_0)^4 = \rho_\xi^4,$$

and, as before, $\rho_\xi = \alpha_2 a_1 - \alpha_3 b_0 = (\alpha_2\beta_3)\,\beta_1^3$, so that

$$v_4 = (\alpha_2\beta_3)\,\beta_1^3.(\alpha_2\gamma_3)\,\gamma_1^3.(\alpha_2\delta_3)\,\delta_1^3.(\alpha_2\varepsilon_3)\,\varepsilon_1^3,$$

and therefore

$$U_4 = (\alpha\beta u)(\alpha\gamma u)(\alpha\delta u)(\alpha\varepsilon u)\,\beta_x^3\gamma_x^3\delta_x^3\varepsilon_x^3$$
$$= \chi_8 x_1^{12} u_1^4 + \ldots . \qquad (28).$$

Next, h_4 is the Hessian of v_4, and therefore

$$h_4 = \tfrac{1}{2}\,(\rho\sigma)^2\rho_\xi^2\sigma_\xi^2$$
$$= \tfrac{1}{2}\,(\alpha_2\beta_3)^2.(\alpha_2\gamma_3)\,\gamma_1^3(\alpha_2\delta_3)\,\delta_1^3.(\beta_2\lambda_3)\,\lambda_1^3\,(\beta_2\mu_3)\,\mu_1^3,$$

hence

$$H_4 = \tfrac{1}{2}\,(\alpha\beta u)^2(\alpha\gamma u)(\alpha\delta u)(\beta\lambda u)(\beta\mu u)\,\gamma_x^3\delta_x^3\lambda_x^3\mu_x^3$$
$$= \chi_9 x_1^{12} u_1^6 + \ldots . \qquad (29).$$

Again, for ϕ_4, which is the cubicovariant of v_4, we have

$$\phi_4 = \tfrac{1}{2}\,(\rho\sigma)^2(\sigma\tau)\,\rho_\xi^2\sigma_\xi\tau_\xi^3$$
$$= \tfrac{1}{2}\,(\alpha_2\beta_3)^2(\beta_2\gamma_3).(\alpha_2\delta_3)\,\delta_1^3\,(\alpha_2\varepsilon_3)\,\varepsilon_1^3.(\beta_2\theta_3)\,\theta_1^3.(\gamma_2\lambda_3)\,\lambda_1^3\,(\gamma_2\mu_3)\,\mu_1^3\,(\gamma_2\nu_3)\,\nu_1^3,$$

and therefore

$$\Phi_4 = \tfrac{1}{2}\,(\alpha\beta u)^2(\beta\gamma u)(\alpha\delta u)(\alpha\varepsilon u)(\beta\theta u)(\gamma\lambda u)(\gamma\mu u)(\gamma\nu u)\,\delta_x^3\varepsilon_x^3\theta_x^3\lambda_x^3\mu_x^3\nu_x^3$$
$$= \chi_{10} x_1^{18} u_1^9 + \ldots . \qquad (30).$$

For i_4, the quadrinvariant of v_4, we have

$$i_4 = \tfrac{1}{2}\,(\rho\sigma)^4,$$

and therefore

$$I_4 = \tfrac{1}{2}\,(\alpha\beta u)^4$$
$$= \chi_{11} u_1^4 + \ldots . \qquad (31).$$

Lastly, j_{24} is the Jacobian of v_3 and v_4, so that

$$j_{24} = \alpha_1^3 (\varepsilon\rho)\, \varepsilon_\xi \rho_\xi^3$$
$$= \alpha_1^3 (\alpha_2\beta_3)\cdot(\alpha_2\gamma_3)\,\gamma_1^3\cdot(\beta_2\lambda_3)\,\lambda_1^3\,(\beta_2\mu_3)\,\mu_1^3\,(\beta_2\nu_3)\,\nu_1^3,$$

and therefore

$$J_{24} = (\alpha\beta u)(\alpha\gamma u)(\beta\lambda u)(\beta\mu u)(\beta\nu u)\,\alpha_x^2\gamma_x^3\lambda_x^3\mu_x^3\nu_x^3$$
$$= \chi_{13}x_1^{14}u_1^5 + \ldots . \qquad (32).$$

It will be seen that in every case the value of $m - p$ agrees with the required value in the earlier table, and we are therefore now justified in enunciating the following theorem:

Every concomitant of the ternary quartic can be algebraically expressed in terms of u_x (the universal concomitant) and of the twelve fundamental concomitants U_1 (the quartic itself); H_2, U_2; U_3, H_3, Φ_3, J_{23}; U_4, H_4, Φ_4, I_4, J_{24}; the leading coefficients of these concomitants are given in (13) and (21), *their order and class in* (22) *to* (32), *and their full expressions can be obtained by* (I) and (II).

41. This fundamental system may be modified—as in §34 for the cubic—in the case of all the concomitants which have Jacobian functions in the leading coefficients. We have

$$\phi_3^2 = f v_3^2 - 4h_3^3,$$

where f is the discriminant of v_3; and thus we may replace ϕ_3 by f, the symbolic form of which is

$$(\alpha_2\beta_3)^2(\gamma_2\delta_3)^2(\alpha_2\gamma_3)(\beta_2\delta_3)\,\alpha_1\beta_1\gamma_1\delta_1,$$

and therefore

$$F = (\alpha\beta u)^2(\gamma\delta u)^2(\alpha\gamma u)(\beta\delta u)\,\alpha_x\beta_x\gamma_x\delta_x$$
$$= (a_3^2d_0^2 - 6a_3b_2c_1d_0 + 4a_3c_1^3 + 4d_0b_2^3 - 3b_2^2c_1^2)\,x_1^4u_1^6 + \ldots \qquad (26')$$

will *replace* Φ_3 *in the system.*

Similarly we might replace j_{23} by p, the intermediate invariant of v_3 and h_3; its symbolic form is

$$\tfrac{1}{2}\,\alpha_1^2\,(\beta_2\gamma_3)^2(\alpha_2\gamma_3)(\alpha_2\beta_3)\,\beta_1\gamma_1,$$

and therefore we have a function

$$\tfrac{1}{2}\,(\beta\gamma u)^2(\alpha\gamma u)(\alpha\beta u)\,\alpha_x^2\beta_x\gamma_x$$
$$= \tfrac{1}{6}\,(\alpha\beta\gamma)(\alpha\beta u)(\alpha\gamma u)(\beta\gamma u)\,u_x\alpha_x\beta_x\gamma_x,$$

so that
$$P = \frac{1}{6}(\alpha\beta\gamma)(\alpha\beta u)(\alpha\gamma u)(\beta\gamma u)\,\alpha_x\beta_x\gamma_x$$
$$= \{c_0(a_3c_1 - b_2^2) - b_1(a_3d_0 - b_2c_1) + a_2(b_2d_0 - c_1^2)\}x_1^3u_1^3 + \ldots \quad (27')$$
will *replace* J_{23} *in the system.*

Next, ϕ_4 being the cubicovariant of v_4, we have
$$\phi_4^2 + j_4v_4^3 - i_4h_4v_4^2 + 4h_4^3 = 0,$$
where j_4 is the cubinvariant of v_4; and so we may replace ϕ_4 by j_4, the symbolical expression of which is
$$\frac{1}{6}(\rho\sigma)^2(\sigma\tau)^2(\rho\tau)^2$$
$$= \frac{1}{6}(\alpha_2\beta_3)^2(\beta_2\gamma_3)^2(\alpha_2\gamma_3)^2,$$
so that
$$J_4 = \frac{1}{6}(\alpha\beta u)^2(\beta\gamma u)^2(\alpha\gamma u)^2$$
$$= (e_0b_2a_4 + 2d_1b_2c_3 - b_2^3 - e_0c_3^2 - d_1^2a_4)\,u_1^6 + \ldots . \quad (30')$$
will *replace* Φ_4 *in the system.*

Lastly, since j_{24} is the Jacobian of v_2 and v_4, and we retain h_2 and h_4, it can be replaced by the second transvectant of v_2 and v_4, the symbolical expression of which is
$$q = \frac{1}{2}\alpha_1^2(\epsilon\rho)^2\rho_\xi^2$$
$$= \frac{1}{2}\alpha_1^2(\alpha_2\beta_3)^2.(\beta_2\gamma_3)\gamma_1^2(\beta_2\delta_3)\delta_1^2,$$
and therefore
$$Q = \frac{1}{2}(\alpha\beta u)^2(\beta\gamma u)(\beta\delta u)\,\alpha_x^2\gamma_x^2\delta_x^2$$
$$= \{(a_2e_0 - 2b_1d_1 + c_0c_2)a_1^2 - 2(a_2d_1 - 2b_1c_2 + b_3c_0)a_1b_0$$
$$+ (a_2c_2 - 2b_1b_3 + c_0a_4)b_0^2\}x_1^6u_1^4 + \ldots . \quad (32')$$
will *replace* J_{24} *in the system.*

42. As illustrations of the general theorem of §40, the following may be taken. It has been shown by Maisano* that all the concomitants of the quartic of the second degree are (l. c., p. 201) $a_x^2b_x^2(abu)^2$, which is effectively H_2 of (22), and $(abu)^4$, which is I_4 of (31); and that among those of the third degree are

* "Sistemi completi dei primi cinque gradi della forma ternaria biquadratica e degl' invarianti, covarianti e contravarianti di sesto grado," Batt. Giorn. di Mat., t. XIX (1881), pp. 198–287.

(l. c., p. 203) $(abu)^2(bcu)^2(cau)^2$, which is effectively J_4 of (30′), and $(abc)^2a_x^2b_x^2c_x^2$, which is $U_2 - U_1H_2$. Another concomitant of this degree is

$$\Theta = a_x b_x^2 c_x^2 (abu)^2(acu) = \{a_1(a_2d_0 - 2b_1c_1 + b_2c_0) - b_0(c_1a_2 - 2b_2c_1 + a_3c_0)\}x_1^6u_1^3 + \ldots,$$

so that
$$\Theta U_2 U_3 = H_2 U_3^2 + H_3 U_2^2 + J_{23}^2.$$

Another is

$$\begin{aligned}\Psi &= (abu)^3(abc)\,c_x^3 \\ &= [a_0(e_0a_4 - 4d_1b_3 + 3c_2^2) + a_1(d_0b_3 - 3c_2c_1 + 3b_2d_1 - a_3e_0) \\ &\qquad - b_0(d_0a_4 - 3b_3c_1 + 3b_2c_2 - a_3d_1)]\,u_1^3x_1^3 + \ldots. \\ &= \psi u_1^3x_1^3 + \ldots,\end{aligned}$$

and it is not difficult to prove that

$$\begin{aligned}&(u_x\Psi - U_1I_4)U_2^3U_3^3U_4^3 \\ &= U_2^3U_3^3\Phi_4 - U_2^3U_4^3\Phi_3 + 3U_2^2(H_4U_3^2 + H_3U_4^2)(U_4J_{23} - U_3J_{24}) + (U_4J_{23} - U_3J_{24})^3.\end{aligned}$$

Lastly, when the tabulated value of $A = \frac{1}{6}(abc)^4$ is taken as calculated by Bernardi, it can be arranged in the form

$$A = \psi - 12p + 3N,$$

where p is the coefficient of (27′), ψ is the coefficient just given and

$$N = e_0a_2^2 - 4d_1a_2b_1 + 2c_2(c_0a_2 + 2b_1^2) - 4b_3c_0b_1 + a_4c_0^2,$$

evidently a simultaneous invariant of v_4 and v_2 and expressible in terms of v_2, h_2, v_4, h_4, i_4, ϕ_4, j_{24}.

The general method of expressing any concomitant in terms of the set, here proved to be complete, is to take its leading coefficient ϑ, which must be a simultaneous concomitant of v_0, v_2, v_3, v_4 and must be expressible in terms of the quantities in (13) and (21). Moreover, since they are binariants, it is sufficient to consider the coefficients of the highest powers of a_1 contained by them; and it is found that in every case ϑ can be arranged as combinations of quantities, which are concomitants in a_1 and b_0. Thus, for instance, Θ above has for its leading coefficient the simultaneous linear covariant called L_1 by Salmon (p. 178); Ψ has a leading coefficient composed of a part v_0i_4 and the simultaneous covariant called L_2 by Salmon (p. 179), and similarly for A.

IV.—*Complete System of Algebraically Independent Concomitants for the n^{tic}.*

43. It is at once evident that all the leading coefficients of the concomitants just obtained for the quartic consist of (i) the algebraically independent invariants and covariants of the binary quadratic, the binary cubic and the binary quartic in a_1 and $-b_0$ as variables; of (ii) the Jacobians of this binary quadratic and binary cubic, and this binary quadratic and binary quartic; and (iii) of the original quantic.

The forms of the characteristic differential equations satisfied by these leading coefficients show that every solution is a concomitant of the simultaneous system of binary quantics formed in the above way; the theory shows that every such solution can be algebraically expressed in terms of the members of the above set.

And the result is true for the general quantic of order n, so that we can now state a complete system of algebraically independent concomitants.

44. First, let $U = a_x^n$ be the quantic which, written explicitly, takes the form

$$(a_0, a_1, \ldots, a_n \between x_1, x_3)^n + \frac{n!}{\overline{n-1}!\,1!} x_2 (b_0, b_1, \ldots, b_{n-1} \between x_1, x_3)^{n-1}$$
$$+ \frac{n!}{\overline{n-2}!\,2!} x_2^2 (c_0, c_1, \ldots, c_{n-2} \between x_1, x_3)^{n-2}$$
$$+ \frac{n!}{\overline{n-3}!\,3!} x_2^3 (d_0, d_1, \ldots, d_{n-3} \between x_1, x_3)^{n-3} + \ldots.$$

We shall represent the leading coefficients in symbolical forms, as in §§33, 40; their explicit forms are obtainable in the same way as the explicit forms of concomitants of binary quantics, and indeed are the same as those binariants when a_1 and $-b_0$ replace the variables.

We have, first

$$U = a_x^n = a_0 x_1^n + \ldots.$$

Next, let

$$v_2 = (c_0, b_1, a_2 \between a_1, -b_0)^2 = \alpha_1^{n-2} (\alpha_2 a_1 - \alpha_3 b_0)^2 = \alpha_1^{n-2} \rho_\xi^2,$$

then the concomitant is

$$U_2 = v_2 x_1^{2n-4} u_1^2 + \ldots;$$

and $h_2 = a_2 c_0 - b_1^2$, the Hessian of v_2, so that the concomitant is

$$H_2 = h_2 x_1^{2n-4} u_1^2 + \ldots,$$

and as usual we have

$$(U_2 - UH_2) \div u_x^2 = \text{Hessian of } U.$$

Next, let

$$v_3 = (d_0, c_1, b_2, a_3 \between a_1, -b_0)^3 = \alpha_1^{n-3}\rho_\xi^3,$$

then the concomitant is

$$U_3 = v_3 x_1^{4n-6} u_1^3 + \ldots . ;$$

and the associated set is given by h_3 (the Hessian) and ϕ_3 (the cubicovariant) of v_3, the concomitants being

$$H_3 = h_3 x_1^{4n-8} u_1^4 + \ldots . ,$$
$$\Phi_3 = \phi_3 x_1^{6n-12} u_1^6 + \ldots . ,$$

and so on.

In general, let

$$v_r = (\ldots . , c_{r-2}, b_{r-1}, a_r \between a_1, -b_0)^r = \alpha^{n-r}\rho_\xi^r,$$

where $\rho_\xi = \alpha_2 a_1 - \alpha_3 b_0$; and we shall suppose ρ, σ, τ to be equivalent symbols. Then it is known from the theory of binary quantics that all the concomitants can be expressed in terms of the following set of binariants of the second and of the third degrees alternately in $\ldots . , c_{r-2}, b_{r-1}, a_r$, viz.

$$\omega(2, r) = \alpha_1^{n-r}\beta_1^{n-r}(\rho\sigma)^2 \rho_\xi^{r-2}\sigma_\xi^{r-2},$$
$$\omega(3, r) = \alpha_1^{n-r}\beta_1^{n-r}\gamma_1^{n-r}(\rho\sigma)^2(\sigma\tau)\rho_\xi^{r-2}\sigma_\xi^{r-3}\tau_\xi^{r-1},$$
$$\omega(4, r) = \alpha_1^{n-r}\beta_1^{n-r}(\rho\sigma)^4 \rho_\xi^{r-4}\sigma_\xi^{r-4},$$
$$\omega(5, r) = \alpha_1^{n-r}\beta_1^{n-r}\gamma_1^{n-r}(\rho\sigma)^4(\sigma\tau)\rho_\xi^{r-4}\sigma_\xi^{r-5}\tau_\xi^{r-1},$$
$$\omega(6, r) = \alpha_1^{n-r}\beta_1^{n-r}(\rho\sigma)^6 \rho_\xi^{r-6}\sigma_\xi^{r-6},$$
$$\omega(7, r) = \alpha_1^{n-r}\beta_1^{n-r}\gamma_1^{n-r}(\rho\sigma)^6(\sigma\tau)\rho_\xi^{r-6}\sigma_\xi^{r-7}\tau_\xi^{r-1},$$

and so on; the symbols ρ_ξ, σ_ξ, τ_ξ in these respectively denote $\alpha_2 a_1 - \alpha_3 b_0$, $\beta_2 a_1 - \beta_3 b_0$, and $\gamma_2 a_1 - \gamma_3 b_0$. The series of functions concludes with the term $\omega(r, r)$, the form of which depends on the evenness or oddness of r.

In order to find the order and the class for each of the concomitants, we must take two separate typical forms, say $\omega(2s, r)$ and $\omega(2s+1, r)$, where r may not be less than $2s$ in the former nor than $2s+1$ in the latter.

For the former we have

$$\omega(2s, r) = \alpha_1^{n-r}\beta_1^{n-r}(\rho\sigma)^{2s}\rho_\xi^{r-2s}\sigma_\xi^{r-2s},$$

when this is changed into a further symbolical form for the concomitant, $(\rho\sigma)$

becomes $(\alpha_2\beta_3)$ and so ultimately comes to be $(\alpha\beta u)$; that is, every power of $(\rho\sigma)$ introduces a unit for the class. Again, ρ_ξ becomes of the form

$$\theta_1^{n-1}(\beta_2\theta_3-\beta_3\theta_2)=\theta_1^{n-1}(\beta_2\theta_3),$$

and so ultimately comes to be $\theta_x^{n-1}(\beta\theta u)$; that is, every factor of the form ρ_ξ introduces a unit for the class and $n-1$ for the order. And α_1^{n-r} ultimately comes to be α_x^{n-r}, and so with β_x. Hence finally, the order is

$$2(n-r)+2(r-2s)(n-1)=2n(r-2s+1)-4(r-s),$$

and the class is $\quad 2s+2(r-2s)=2(r-s)$,

and therefore the concomitant is

$$W_{2s,\,r}=\omega(2s,\,r)\,x_1^{2n(r-2s+1)-4(r-s)}u_1^{2(r-s)}+\ldots.$$

Similarly for $\omega(2s+1,\,r)$, the symbol for which is

$$\alpha_1^{n-r}\beta_1^{n-r}\gamma_1^{n-r}(\rho\sigma)^{2s}(\sigma\tau)\,\rho_\xi^{r-2s}\sigma_\xi^{r-2s-1}\tau_\xi^{r-1},$$

we have

$$W_{2s+1,\,r}=\omega(2s+1,\,r)\,x_1^{n(3r-4s+1)-2(3r-2s-1)}u_1^{3r-2s-1}+\ldots.$$

For this class of the complete set of the concomitants given by $W_{\mu,\,r}$, the values of μ, for a given value of r, are $0,\,2,\,3,\,\ldots,\,r$ and $W_{0,\,r}$ has for its leading coefficient v_r, being

$$=v_r x_1^{n(r+1)-2r}u_1^r+\ldots;$$

and the values of r are $2,\,3,\,\ldots,\,n$. Thus the total number of concomitants in this division is $\frac{1}{2}n(n+1)-1$.

45. Next, for the class of concomitants whose coefficients are the algebraically independent Jacobians of $v_2,\,v_3,\,\ldots,\,v_n$, we take $j_{2,\,3},\,j_{2,\,4},\,j_{2,\,5},\,\ldots,\,j_{2,\,n}$. Evidently

$$j_{2,\,r}=\alpha_1^{n-2}\beta_1^{n-r}(\rho\sigma)\,\rho_\xi\sigma_\xi^{r-1},$$

where $\rho_\xi=\alpha_2a_1-\alpha_3b_0$, $\sigma_\xi=\beta_2a_1-\beta_3b_0$; and the concomitant is

$$J_{2,\,r}=j_{2,\,r}\,x_1^{n(r+2)-2r-2}u_1^{r+1}+\ldots.$$

The values of r are $3,\,4,\,\ldots,\,n$, and the total number in this class of concomitants is therefore $n-2$.

Hence the *total number of concomitants* is

$$\begin{aligned} &1\text{, for the original quantic,}\\ &+\tfrac{1}{2}n(n+1)-1 \text{ for those in the first of the classes,}\\ &+n-2 \quad \text{" " " second " " .}\end{aligned}$$

i. e. the total number is $\frac{1}{2}(n+4)(n-1)$, agreeing with the former result.

These concomitants are algebraically independent of one another, and every concomitant of the quantic can be algebraically expressed in terms of them.

V.—*System of Two Quadratics.*

46. The two quadratics may be taken in the forms

$$a_0x_1^2+2b_0x_1x_2+2a_1x_1x_3+c_0x_2^2+2b_1x_2x_3+a_2x_3^2,$$
$$a_0'x_1^2+2b_0'x_1x_2+2a_1'x_1x_3+c_0'x_2^2+2b_1'x_2x_3+a_2'x_3^2;$$

the characteristic equations are

$$D_1+D_1'=a_1\frac{\partial}{\partial b_0}+a_2\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial c_0}+a_1'\frac{\partial}{\partial b_0'}+a_2'\frac{\partial}{\partial b_1'}+2b_1'\frac{\partial}{\partial c_0'},$$
$$\Delta=D_6+D_6'=b_0\frac{\partial}{\partial a_1}+c_0\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial a_2}+b_0'\frac{\partial}{\partial a_1'}+c_0'\frac{\partial}{\partial b_1'}+2b_1'\frac{\partial}{\partial a_2'}.$$

There are twelve coefficients in all; there will therefore be eleven equations subsidiary to, and eleven independent solutions of, $D_1+D_1'=0$; and ultimately there will be (§§18 and 35) nine independent solutions common to the two equations.

From the form of the characteristic equations, it at once follows that they are the *simultaneous concomitants of two binary quadratics, the literal coefficients of which are c_0, b_1, a_2, and c_0', b_1', a_2', and that the variables of the concomitants are two sets, viz. a_1 and $-b_0$, a_1' and $-b_0'$.*

47. The subsidiary equations for $D_1+D_1'=0$ are

$$\frac{da_0}{0}=\frac{da_1}{0}=\frac{da_2}{0}=\frac{da_0'}{0}=\frac{da_1'}{0}=\frac{da_2'}{0}=\frac{db_0}{a_1}=\frac{db_1}{a_2}=\frac{dc_0}{2b_1}=\frac{db_0'}{a_1'}=\frac{db_1'}{a_2'}=\frac{dc_0'}{2b_1'},$$

of which six integrals are immediately given by

$$\theta_0=a_0,\quad \theta_1=a_1,\quad \theta_2=a_2;$$
$$\theta_0'=a_0',\quad \theta_1'=a_1',\quad \theta_2'=a_2',$$

and we may take either θ_1 or θ_1' as a "variable of reference."

The full system of equations, subsidiary to the solution of $\Delta = 0$ in functional combinations of the solutions of $D_1 + D_1' = 0$, are for the alternative variables of reference:

$$\left.\begin{array}{l} \theta_1\Delta\theta_2 - 2\theta_2\Delta\theta_1 = 2\theta_3 \\ \theta_1\Delta\theta_3 - \theta_3\Delta\theta_1 = \theta_4 \\ \theta_1\Delta\theta_1' - \theta_1'\Delta\theta_1 = \phi \\ \theta_1\Delta\theta_2' - 2\theta_2'\Delta\theta_1 = 2\psi_3 \\ \theta_1\Delta\psi_3 - \psi_3\Delta\theta_1 = \psi_4 \\ \hline \theta_1\Delta\theta_3' - \theta_3'\Delta\theta_1 = \mu_4 \\ \theta_1\Delta\chi_3 - \chi_3\Delta\theta_1 = \lambda_4 \end{array}\right\}, \qquad \left.\begin{array}{l} \theta_1'\Delta\theta_2' - 2\theta_2'\Delta\theta_1' = 2\theta_3' \\ \theta_1'\Delta\theta_3' - \theta_3'\Delta\theta_1' = \theta_4' \\ \theta_1'\Delta\theta_1 - \theta_1\Delta\theta_1' = -\phi \\ \theta_1'\Delta\theta_2 - 2\theta_2\Delta\theta_1' = 2\chi_3 \\ \theta_1'\Delta\chi_3 - \chi_3\Delta\theta_1' = \chi_4 \\ \hline \theta_1'\Delta\theta_3 - \theta_3\Delta\theta_1' = \lambda_4 \\ \theta_1'\Delta\psi_3 - \psi_3\Delta\theta_1' = \mu_4 \end{array}\right\}$$

where the eleven quantities defined by the equations

$$\left.\begin{array}{l} \theta_3 = b_1a_1 - a_2b_0 \\ \chi_3 = b_1a_1' - a_2b_0' \end{array}\right\}, \quad \left.\begin{array}{l} \psi_3 = b_1'a_1 - a_2'b_0 \\ \theta_3' = b_1'a_1' - a_2'b_0' \end{array}\right\}, \quad \phi = a_1b_0' - a_1'b_0,$$

$$\left.\begin{array}{l} \theta_4 = c_0a_1^2 - 2b_1a_1b_0 + a_2b_0^2 \\ \lambda_4 = c_0a_1a_1' - b_1(a_1b_0' + a_1'b_0) + a_2b_0b_0' \\ \chi_4 = c_0a_1'^2 - 2b_1a_1'b_0' + a_2b_0'^2 \end{array}\right\}, \quad \left.\begin{array}{l} \psi_4 = c_0'a_1^2 - 2b_1'a_1b_0 + a_2'b_0^2 \\ \mu_4 = c_0'a_1a_1' - b_1'(a_1b_0' + a_1'b_0) + a_2'b_0b_0' \\ \theta_4' = c_0'a_1'^2 - 2b_1'a_1'b_0' + a_2'b_0'^2 \end{array}\right\},$$

all are solutions of $D_1 + D_1' = 0$. Further, the quantities $\theta_4, \psi_4; \theta_4', \chi_4; \phi; \lambda_4, \mu_4$, are solutions also of $\Delta = 0$.

For each of the variables of reference, the first five of the equations of the set are sufficient to give all the equations, subsidiary to $\Delta = 0$ and necessary for the derivation of solutions additional to those already obtained.

Taking θ_1 as the variable of reference, we have common solutions of the two characteristic equations given by

$$\theta_0, \theta_0'; \theta_4, \psi_4; \phi;$$

and four more are necessary, given by the solutions of the first five equations in the first bracket of modified equations subsidiary to $\Delta = 0$. If, then, we substitute

$$\left.\begin{array}{l} \dfrac{\theta_2}{\theta_1^2} = p \\ \dfrac{\theta_2'}{\theta_1'^2} = p' \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{\theta_3}{\theta_1} = q \\ \dfrac{\theta_3'}{\theta_1'} = q' \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{\theta_2'}{\theta_1^2} = r \\ \dfrac{\theta_2}{\theta_1'^2} = r' \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{\psi_3}{\theta_1} = \rho \\ \dfrac{\chi_3}{\theta_1'} = \rho' \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{\chi_3}{\theta_1} = \sigma \\ \dfrac{\psi_3}{\theta_1'} = \sigma' \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{\theta_3'}{\theta_1} = s \\ \dfrac{\theta_3}{\theta_1'} = s' \end{array}\right\}, \quad \dfrac{\theta_1'}{\theta_1} = \varepsilon,$$

then the equations come to be

$$\left.\begin{aligned} \theta_1^2\Delta p &= 2q \\ \theta_1^2\Delta q &= \theta_4 \\ \theta_1^2\Delta \varepsilon &= \phi \\ \theta_1^2\Delta r &= 2\rho \\ \theta_1^2\Delta \rho &= \psi_4 \\ \hline \theta_1^2\Delta s &= \mu_4 \\ \theta_1^2\Delta \sigma &= \lambda_4 \end{aligned}\right\}, \qquad \left.\begin{aligned} \theta_1'^2\Delta p' &= 2q' \\ \theta_1'^2\Delta q' &= \theta_4' \\ \theta_1'^2\Delta \varepsilon^{-1} &= -\phi \\ \theta_1'^2\Delta r' &= 2\rho' \\ \theta_1'^2\Delta \rho' &= \chi_4 \\ \hline \theta_1'^2\Delta s' &= \lambda_4 \\ \theta_1'^2\Delta \sigma' &= \mu_4 \end{aligned}\right\},$$

and what we wish are four independent solutions of the first five equations in the former of these brackets. Such solutions are

$$\begin{aligned} \vartheta_2 &= p\theta_4 - q^2 = a_2c_0 - b_1^2, \\ B &= \varepsilon\theta_4 - q\phi = \lambda_4, \\ \vartheta_2' &= r\psi_4 - \rho^2 = a_2'c_0' - b_1'^2, \\ C &= \varepsilon\psi_4 - \rho\phi = \mu_4; \end{aligned}$$

and λ_4 and μ_4 are the respective Jacobians of ϕ and θ_4 (with a_1 and b_0 as variables) and of ϕ and ψ_4 (with a_1 and b_0 as variables).

Hence it follows that *every common solution of the two characteristic equations can be expressed in terms of the nine common solutions*

$$\theta_0,\ \theta_4,\ \vartheta_2;\ \theta_0',\ \psi_4,\ \vartheta_2';\ \phi,\ \lambda_4,\ \mu_4.$$

48. If we take the first five of the modified equations in the second bracket, we find the four new solutions to be

$$\begin{aligned} \vartheta_2 &= r'\chi_4 - \rho'^2 = a_2c_0 - b_1^2, \\ B &= q'\phi + \theta_4'\varepsilon^{-1} = \lambda_4, \\ \vartheta_2' &= p'\theta_4' - q'^2 = a_2'c_0' - b_1'^2, \\ C &= \rho'\phi + \chi_4\varepsilon^{-1} = \mu_4; \end{aligned}$$

and *every solution can be expressed in terms of the set*

$$\theta_0,\ \vartheta_2,\ \chi_4;\ \theta_0',\ \theta_4',\ \vartheta_2';\ \phi,\ \lambda_4,\ \mu_4.$$

49. Other solutions of the system of equations are

$$f_{12} = a_2c_0' + a_2'c_0 - 2b_1b_1',$$

an intermediate between ϑ_2 and ϑ_2';

$$\begin{aligned} g &= \rho\theta_4 - q\psi_4 = (b_1'c_0 - b_1c_0')\,a_1^2 - a_1b_0\,(a_2'c_0 - a_2c_0') + b_0^2\,(a_2'b_1 - a_2b_1'), \\ g' &= -\rho'\theta_4' + q'\chi_4 = (b_1'c_0 - b_1c_0')\,a_1'^2 - a_1'b_0'\,(a_2'c_0 - a_2c_0') + b_0'^2\,(a_2'b_1 - a_2b_1'), \end{aligned}$$

the former a Jacobian of θ_4 and ψ_4 (in a_1 and b_0 as variables), the latter a Jacobian of θ_4' and χ_4 (in a_1' and b_0' as variables);

$$g_{12} = s\theta_4 - q\mu_4 = (b_1'c_0 - b_1c_0')\,a_1a_1' - a_1b_0'\,(a_2'c_0 - b_1b_1') - a_1'b_0\,(b_1b_1' - a_2c_0') + b_0b_0'\,(a_2'b_1 - a_2b_1'),$$

an intermediary between g and g', and a Jacobian of θ_4 and μ_4.

These four are the most important of the solutions, and they will be used in connection with the fundamental system to be made symmetrical later.

Other solutions—the simplest in form—are as follows: they should be expressible in terms of the fundamental set, and the verification of this leads to the values given for them. The left-hand sides of the equations give the solutions, the right-hand their values:

$$\begin{aligned} s\psi_4 - \rho\mu_4 &= -\phi\vartheta_2', & s'\chi_4 - \rho'\lambda_4 &= \phi\vartheta_2; \\ q\lambda_4 - \sigma\theta_4 &= \phi\vartheta_2, & q'\mu_4 - \sigma'\theta_4' &= -\phi\vartheta_2'; \\ s\theta_4 - q\mu_4 &= g_{12}, & s'\theta_4' - q'\lambda_4 &= -g_{12}; \\ \rho\theta_4 - q\psi_4 &= g\,, & \rho'\theta_4' - q'\chi_4 &= -g'\,; \\ \varepsilon\theta_4 - q\phi &= \lambda_4, & \frac{1}{\varepsilon}\theta_4' + q'\phi &= \mu_4\,; \\ \varepsilon\psi_4 - \rho\phi &= \mu_4, & \frac{1}{\varepsilon}\chi_4 + \rho'\phi &= \lambda_4; \\ \varepsilon\mu_4 - s\phi &= \theta_4, & \frac{1}{\varepsilon}\lambda_4 + s'\phi &= \theta_4'; \\ \varepsilon\lambda_4 - \sigma\phi &= \chi_4, & \frac{1}{\varepsilon}\mu_4 + \sigma'\phi &= \psi_4; \\ \rho\lambda_4 - \sigma\psi_4 &= g_{12} + \phi f_{12}, & \rho'\mu_4 - \sigma'\chi_4 &= -g_{12} - \phi f_{12}; \\ s\lambda_4 - \sigma\mu_4 &= g\,, & s'\mu_4 - \sigma'\lambda_4 &= -g'. \end{aligned}$$

And the equations which express the values of the quantities g, g', g_{12}, f_{12}, ... in terms of the fundamental systems are

$$\begin{aligned} \theta_4'\psi_4 &= \mu_4^2 + \phi^2\vartheta_2', \\ \theta_4\chi_4 &= \lambda_4^2 + \phi^2\vartheta_2, \\ g\phi &= \lambda_4\psi_4 - \mu_4\theta_4, \\ g'\phi &= \mu_4\chi_4 - \lambda_4\theta_4', \\ &\left.\begin{aligned} g^2 &= -\vartheta_2\psi_4^2 + f_{12}\theta_4\psi_4 - \vartheta_2'\theta_4^2 \\ g'^2 &= -\vartheta_2'\chi_4^2 + f_{12}\theta_4'\chi_4 - \vartheta_2\theta_4'^2 \end{aligned}\right\}; \\ &\left.\begin{aligned} g_{12}\psi_4 &= \mu_4 g - \theta_4\vartheta_2'\phi \\ g_{12}\chi_4 &= \lambda_4 g' - \theta_4'\vartheta_2\phi \end{aligned}\right\}. \end{aligned}$$

It may be remarked that a form more directly intermediate between g and g' is given by $g_{12} + \frac{1}{2} f_{12}\phi$, the value of which is

$$(b_1'c_0 - b_1c_0')\,a_1a_1' - \frac{1}{2}(a_2'c_0 - a_2c_0')(a_1b_0' + a_1'b_0) + (a_2'b_1 - a_2b_1')\,b_0b_0',$$

which with similar forms will be adopted for the system in the case of three quadratics. The form g_{12} adopted in the present system is directly connected with one of Gordan's concomitants, and the corresponding concomitant has its order and its class each greater by unity than those of the present g_{12}.

50. The fundamental system can be modified so as to be symmetrical with regard to the two quantics. We have seen that

$$\theta_4'\psi_4 = \mu_4^2 + \phi^2\vartheta_2',$$
$$\theta_4\chi_4 = \lambda_4^2 + \phi^2\vartheta_2,$$

so that in the first fundamental system we can replace λ_4 and μ_4 by χ_4 and θ_4' respectively, and in the second by θ_4 and ψ_4 respectively. The two systems are the same, and it thus follows that *every common solution can be expressed in terms of* $\theta_0,\ \theta_0';\ \phi;\ \vartheta_2,\ \vartheta_2';\ \theta_4,\ \psi_4,\ \chi_4,\ \theta_4'$.

51. It is now necessary to determine the order and the grade of each of the concomitants determined by these leading coefficients. It is easy to show that

$$\begin{aligned}
U &= \theta_0 x_1^2 + \ldots\ldots, \\
U' &= \theta_0' x_1^2 + \ldots\ldots, \\
\Phi &= \phi x_1^2 u_1 + \ldots\ldots, \\
\Theta_2 &= \vartheta_2 u_1^2 + \ldots\ldots, \\
\Theta_2' &= \vartheta_2' u_1^2 + \ldots\ldots, \\
\Theta_4 &= \theta_4 x_1^2 u_1^2 + \ldots\ldots, \\
\Psi_4 &= \psi_4 x_1^2 u_1^2 + \ldots\ldots, \\
X_4 &= \chi_4 x_1^2 u_1^2 + \ldots\ldots, \\
\Theta_4' &= \theta_4' x_1^2 u_1^2 + \ldots\ldots,
\end{aligned}$$

and therefore *every simultaneous concomitant of the two quadratics can be expressed algebraically in terms of* $U,\ U',\ \Phi,\ \Theta_2,\ \Theta_2',\ \Theta_4,\ \Psi_4,\ X_4,\ \Theta_4'$.

In addition to these nine, it is convenient to have other six, the leading coefficients of which are respectively λ_4, μ_4; g, g', g_{12}, and f_{12}. It is easy to determine their order and class; the concomitants are

$$
\begin{aligned}
\Lambda_4 &= \lambda_4 x_1^2 u_1^2 + \ldots\ldots, \\
\mathrm{M}_4 &= \mu_4 x_1^2 u_1^2 + \ldots\ldots, \\
G &= g x_1^2 u_1^3 + \ldots\ldots, \\
G' &= g' x_1^2 u_1^3 + \ldots\ldots, \\
G_{12} &= g_{12} x_1 u_1^2 + \ldots\ldots, \\
F_{12} &= f_{12} u_1^3 \;\; + \ldots\ldots,
\end{aligned}
$$

and the equations which give the values of these in terms of the members of the fundamental system are

$$
\begin{aligned}
\mathrm{M}_4^2 &= \Theta_4'\Psi_4 - \Phi^2\Theta_2', \\
\Lambda_4^2 &= \Theta_4 X_4 - \Phi^2\Theta_2, \\
G\Phi &= \mathrm{M}_4\Theta_4 - \Lambda_4\Psi_4, \\
G'\Phi &= \Lambda_4\Theta_4' - \mathrm{M}_4 X_4, \\
&\left.\begin{aligned}
\Theta_4\Psi_4 F_{12} &= G^2 \;\; + \Theta_2\Psi_4^2 + \Theta_2'\Theta_4^2 \\
\Theta_4' X_4 F_{12} &= G'^2 \;\; + \Theta_2\Theta_4'^2 + \Theta_2' X_4^2
\end{aligned}\right\}, \\
&\left.\begin{aligned}
u_x\Psi_4 G_{12} &= \mathrm{M}_4 G - \Theta_4\Theta_2'\Phi \\
u_x X_4 G_{12} &= \Lambda_4 G' - \Theta_4'\Theta_2\Phi
\end{aligned}\right\}.
\end{aligned}
$$

The six concomitants Λ_4, M_4, G, G', G_{12}, F_{12} may be used as subsidiary to the symmetrical set, it being understood that in expressions they represent the foregoing combinations of the members of that set.

52. Now Gordan has shown* that the number of asyzygetic concomitants of a system of two ternary quadratics is 20, and he has given (l. c.) the symbolical expressions for them. From the foregoing theory it follows that each of them must be expressible in terms of the set of nine above obtained; the expressions I find to be as follows:

---

* Clebsch, "Vorlesungen über Geometrie" (Lindemann), pp. 288-291 and note on p. 290.

$$\left.\begin{aligned} f &= U, \\ F_{11} &= 2\Theta_2, \\ u_x^2 A_{111} &= 6(\Theta_2 U - \Theta_4), \\ u_x^2 A_{112} &= 2(U'\Theta_2 - 2\Lambda_4 - \Psi_4 + UF_{12}), \\ u_x B_1 &= 2(U'\Theta_2 - \Lambda_4), \end{aligned}\right\}.$$

$$\left.\begin{aligned} f' &= U', \\ F_{22} &= 2\Theta_2', \\ u_x^2 A_{222} &= 6(\Theta_2' U' - \Theta_4'), \\ u_x^2 A_{221} &= 2(U\Theta_2' - 2\mathrm{M}_4 - X_4 + U'F_{12}), \\ u_x B_2 &= 2(U\Theta_2' - \mathrm{M}_4), \end{aligned}\right\}$$

$$\left.\begin{aligned} N &= -\Phi, \\ u_x C_1 &= 2(G - \Theta_2\Phi), \\ u_x C_2 &= 2(G' - \Theta_2'\Phi), \\ u_x^2 D &= 4(\Theta_2 G' + \Theta_2' G - 2\Theta_2\Theta_2'\Phi - u_x F_{12} G_{12}), \end{aligned}\right\}$$

$$\left.\begin{aligned} N &= 4G_{12}, \\ u_x^2 \Gamma_1 &= 4(\Phi\Lambda_4 - U'G + u_x U G_{12}), \\ u_x^2 \Gamma_2 &= 4(\Phi\mathrm{M}_4 - UG' + u_x U' G_{12}), \\ u_x^3 \Delta &= 4(\Phi^3 - U'\Phi\Psi_4 - U\Phi X_4 + U'\Phi\Lambda_4 + U\Phi M_4 \\ &\qquad - U^2 G' - U'^2 G + UU'\Phi F_{12} + 2u_x UU' G_{12}), \end{aligned}\right\}$$

$$\left.\begin{aligned} f_{12} &= F_{12}, \\ u_x^2 \Phi_{12} &= 4(\Phi^2 - UX_4 - U'\Psi_4 + UU'F_{12}), \end{aligned}\right\}$$

the symbols on the left-hand sides being those used by Gordan. From these relations it is easy to deduce the equations

$$\begin{aligned} u_x D &= F_{11} C_2 + F_{22} C_1 - Nf_{12}, \\ u_x \Delta &= f'\Gamma_1 + f\Gamma_2 - N\Phi_{12}, \end{aligned}$$

subsisting among Gordan's concomitants.

VI.—*System of Three Quadratics.*

53. They may be taken in the forms

$$\begin{aligned} &a_0 x_1^2 + 2b_0 x_1 x_2 + 2a_1 x_1 x_3 + c_0 x_2^2 + 2b_1 x_2 x_3 + a_2 x_3^2, \\ &a_0' x_1^2 + 2b_0' x_1 x_2 + 2a_1' x_1 x_3 + c_0' x_2^2 + 2b_1' x_2 x_3 + a_2' x_3^2, \\ &a_0'' x_1^2 + 2b_0'' x_1 x_2 + 2a_1'' x_1 x_3 + c_0'' x_2^2 + 2b_1'' x_2 x_3 + a_2'' x_3^2; \end{aligned}$$

the characteristic equations are

$$D_1 + D_1' + D_1'' = 0 \text{ and } \Delta = D_6 + D_6' + D_6'' = 0.$$

There are eighteen coefficients in all; there will therefore be seventeen equations subsidiary to the first of the characteristic equations, requiring seventeen independent integrals. The number of modified Δ-equations is sixteen, and there will therefore be fifteen solutions independent of one another and common to the two characteristic equations.

Hence *all the simultaneous concomitants can be expressed in terms of fifteen concomitants.*

In what follows, only the results are given; they are derived by algebraical analysis similar to what has preceded, and the solutions evidently maintain the preceding analogy to binariants.

In forming the equations, the following quantities occur:

$$\left.\begin{aligned} \theta_0 &= a_0 \\ \theta_1 &= a_1 \\ \theta_2 &= a_1 \end{aligned}\right\}, \qquad \left.\begin{aligned} \theta_0' &= a_0' \\ \theta_1' &= a_1' \\ \theta_2' &= a_2' \end{aligned}\right\}, \qquad \left.\begin{aligned} \theta_0'' &= a_0'' \\ \theta_1'' &= a_1'' \\ \theta_2'' &= a_2'' \end{aligned}\right\};$$

$$\left.\begin{aligned} \theta_3 &= b_1 a_1 - a_2 b_0 \\ \psi_3 &= b_1' a_1 - a_2' b_0 \\ \xi_3 &= b_1'' a_1 - a_2'' b_0 \end{aligned}\right\}, \qquad \left.\begin{aligned} \chi_3 &= b_1 a_1' - a_2 b_0' \\ \theta_3' &= b_1' a_1' - a_2' b_0' \\ \xi_3' &= b_1'' a_1' - a_2'' b_0' \end{aligned}\right\}, \qquad \left.\begin{aligned} \eta_3 &= b_1 a_1'' - a_2 b_0'' \\ \eta_3' &= b_1' a_1'' - a_2' b_0'' \\ \theta_3'' &= b_1'' a_1'' - a_2'' b_0'' \end{aligned}\right\}.$$

Of these quantities, only θ_0, θ_0', θ_0'' are solutions of the equations. The modified Δ-equations are constructed for the three possible cases, according as θ_1, θ_1', or θ_1'' is taken as the variable of reference.

The further quantities here following also occur; they all are simultaneous solutions of the two characteristic equations:

$$\left.\begin{aligned} \theta_4 &= (c_0, b_1, a_2 \between a_1, -b_0)^2 \\ \chi_4 &= (c_0, b_1, a_2 \between a_1', -b_0')^2 \\ \eta_4 &= (c_0, b_1, a_2 \between a_1'', -b_0'')^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \lambda_4 &= (c_0, b_1, a_2 \between a_1, -b_0 \between a_1', -b_0') \\ \lambda_4' &= (c_0, b_1, a_2 \between a_1, -b_0 \between a_1'', -b_0'') \\ \lambda_4'' &= (c_0, b_1, a_2 \between a_1', -b_0' \between a_1'', -b_0'') \end{aligned}\right\};$$

$$\left.\begin{aligned} \psi_4 &= (c_0', b_1', a_2' \between a_1, -b_0)^2 \\ \theta_4' &= (c_0', b_1', a_2' \between a_1', -b_0')^2 \\ \eta_4' &= (c_0', b_1', a_2' \between a_1'', -b_0'')^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \mu_4 &= (c_0', b_1', a_2' \between a_1, -b_0 \between a_1', -b_0') \\ \mu_4' &= (c_0', b_1', a_2' \between a_1, -b_0 \between a_1'', -b_0'') \\ \mu_4'' &= (c_0', b_1', a_2' \between a_1', -b_0' \between a_1'', -b_0'') \end{aligned}\right\};$$

$$\left.\begin{aligned} \xi_4 &= (c_0'', b_1'', a_2'' \between a_1, -b_0)^2 \\ \xi_4' &= (c_0'', b_1'', a_2'' \between a_1', -b_0')^2 \\ \theta_4'' &= (c_0'', b_1'', a_2'' \between a_1'', -b_0'')^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \nu_4 &= (c_0'', b_1'', a_2'' \between a_1, -b_0 \between a_1', -b_0') \\ \nu_4' &= (c_0'', b_1'', a_2'' \between a_1, -b_0 \between a_1'', -b_0'') \\ \nu_4'' &= (c_0'', b_1'', a_2'' \between a_1', -b_0' \between a_1'', -b_0'') \end{aligned}\right\};$$

where in the right-hand column λ_4 denotes $c_0 a_1 a_1' - b_1(a_1 b_0' + a_1' b_0) + a_2 b_0 b_0'$, and similarly for the others.

54. Taking first the equations independent of one another and formed with θ_1 as the variable of reference, we have

$$\left.\begin{aligned}\theta_1\Delta\theta_2 - 2\theta_2\Delta\theta_1 &= 2\theta_3\\ \theta_1\Delta\theta_3 - \theta_3\Delta\theta_1 &= \theta_4\end{aligned}\right\}, \quad \left.\begin{aligned}\theta_1\Delta\theta_1' - \theta_1'\Delta\theta_1 &= \phi\\ \theta_1\Delta\theta_2' - 2\theta_2'\Delta\theta_1 &= 2\psi_3\\ \theta_1\Delta\psi_2 - \psi_3\Delta\theta_1 &= \psi_4\end{aligned}\right\}, \quad \left.\begin{aligned}\theta_1\Delta\theta_1'' - \theta_1''\Delta\theta_1 &= -\phi'\\ \theta_1\Delta\theta_2'' - 2\theta_2''\Delta\theta_1 &= 2\xi_3\\ \theta_1\Delta\xi_3 - \xi_3\Delta\theta_1 &= \xi_4\end{aligned}\right\},$$

where $\phi = a_1 b_0' - a_1' b_0$, $\phi' = a_1'' b_0 - a_1 b_0''$ are simultaneous solutions of the two characteristic equations.

A set of independent solutions of these equations—necessarily seven in number to make up the required fifteen, for we already have $\theta_0, \theta_0', \theta_0''$; θ_4, ψ_4, ξ_4; ϕ and ϕ'—is

$$\left.\begin{aligned}(\theta_2\theta_4 - \theta_3^2) \div \theta_1^2 &= \vartheta_2 = a_2 c_0 - b_1^2\\ (\theta_2'\psi_4 - \psi_3^2) \div \theta_1^2 &= \vartheta_2' = a_2' c_0' - b_1'^2\\ (\theta_2''\xi_4 - \xi_3^2) \div \theta_1^2 &= \vartheta_2'' = a_2'' c_0'' - b_1''^2\end{aligned}\right\},$$

$$\left.\begin{aligned}(\theta_1'\theta_4 - \theta_3\phi) \div \theta_1 &= \lambda_4\\ (\theta_1'\psi_4 - \psi_3\phi) \div \theta_1 &= \mu_4\\ (\theta_1'\xi_4 - \xi_3\phi) \div \theta_1 &= \nu_4\end{aligned}\right\},$$

$$(-\theta_1'\phi' - \theta_1''\phi) \div \theta_1 = \phi'',$$

where the quantities ϕ are

$$\left.\begin{aligned}\phi &= a_1 b_0' - a_1' b_0\\ \phi' &= a_1'' b_0 - a_1 b_0''\\ \phi'' &= a_1' b_0'' - a_1'' b_0'\end{aligned}\right\}.$$

Hence it follows that every simultaneous solution of the two equations can be expressed in terms of the fifteen independent solutions already obtained, viz. $\theta_0, \theta_0', \theta_0''$; θ_4, ψ_4, ξ_4; ϕ, ϕ', ϕ''; $\vartheta_2, \vartheta_2', \vartheta_2''$; λ_4, μ_4, ν_4.

As this set of fifteen is not symmetrical with regard to the three quadratics, it will be replaced immediately by an equivalent set of fifteen independent solutions which shall be symmetrical.

55. Taking now the equations for each of the three possible variables of reference, we find that the foregoing set of eight is increased when all the quantities which arise in the other sets are treated similarly with θ_1 as the variable; the new equations thus obtained are, of course, not independent equations as they can be derived from the eight, but it is convenient so to increase the set in

order to have them complete in form. Introducing quantities defined by the equations

$$\left.\begin{array}{l} \theta_2 = \theta_1^2 p = \theta_1'^2 r' = \theta_1''^2 t'' \\ \theta_2' = \theta_1^2 r = \theta_1'^2 p' = \theta_1''^2 n'' \\ \theta_2'' = \theta_1^2 t = \theta_1'^2 n' = \theta_1''^2 p'' \end{array}\right\}, \quad \left.\begin{array}{l} \theta_3 = \theta_1 q = \theta_1' s' = \theta_1'' l'' \\ \theta_3' = \theta_1 s = \theta_1' q' = \theta_1'' k'' \\ \theta_3'' = \theta_1 l = \theta_1' k = \theta_1'' q'' \end{array}\right\}, \quad \left.\begin{array}{l} \theta_1 \qquad\qquad = \theta_1' \varepsilon' = \theta_1'' \gamma'' \\ \theta_1' = \theta_1 \varepsilon \qquad\quad = \theta_1'' \delta'' \\ \theta_1'' = \theta_1 \gamma = \theta_1' \delta' \end{array}\right\},$$

$$\left.\begin{array}{l} \psi_3 = \theta_1 \rho = \theta_1' \omega' = \theta_1'' \pi'' \\ \xi_3' = \theta_1 \omega = \theta_1' \rho' = \theta_1'' \varkappa'' \\ \eta_3 = \theta_1 \pi = \theta_1' \varkappa' = \theta_1'' \rho'' \end{array}\right\}, \quad \left.\begin{array}{l} \chi_3 = \theta_1 \sigma = \theta_1' \iota' = \theta_1'' \tau'' \\ \eta_3' = \theta_1 \iota = \theta_1' \sigma' = \theta_1'' v'' \\ \xi_3 = \theta_1 \tau = \theta_1' v' = \theta_1'' \sigma'' \end{array}\right\},$$

the three completed sets of modified Δ-equations are

$\theta_1^2 \Delta = \nabla$	$\theta_1'^2 \Delta = \nabla'$	$\theta_1''^2 \Delta = \nabla''$
$\nabla p = 2q$	$\nabla' p' = 2q'$	$\nabla'' p'' = 2q''$
$\nabla q = \theta_4$	$\nabla' q' = \theta_4'$	$\nabla'' q'' = \theta_4''$
$\nabla \varepsilon = \phi$	$\nabla' \varepsilon' = -\phi$	$\nabla'' \gamma'' = \phi'$
$\nabla \gamma = -\phi'$	$\nabla' \delta' = \phi''$	$\nabla'' \delta'' = -\phi''$
$\nabla r = 2\rho$	$\nabla' r' = 2\iota'$	$\nabla'' t'' = 2\rho''$
$\nabla \rho = \psi_4$	$\nabla' \iota' = \chi_4$	$\nabla'' \rho'' = \eta_4$
$\nabla t = 2\tau$	$\nabla' n' = 2\rho'$	$\nabla'' n'' = 2v''$
$\nabla \tau = \xi_4$	$\nabla' \rho' = \xi_4'$	$\nabla'' v'' = \eta_4'$
$\nabla \sigma = \lambda_4$	$\nabla' \varkappa' = \lambda_4''$	$\nabla'' \tau'' = \lambda_4''$
$\nabla \pi = \lambda_4'$	$\nabla' s' = \lambda_4$	$\nabla'' l'' = \lambda_4'$
$\nabla \iota = \mu_4'$	$\nabla' \omega' = \mu_4$	$\nabla'' \pi'' = \mu_4'$
$\nabla s = \mu_4$	$\nabla' \sigma' = \mu_4''$	$\nabla'' k'' = \mu_4''$
$\nabla \omega = \nu_4$	$\nabla' v' = \nu_4$	$\nabla'' \sigma'' = \nu_4'$
$\nabla l = \nu_4'$	$\nabla' k = \nu_4''$	$\nabla'' \varkappa'' = \nu_4''$

in each of which sets the first eight are the independent equations for that set.*

56. We first modify the algebraically complete system of solutions so that it may become symmetrical with regard to the three quantics. We have

$$\begin{aligned} \theta_4\chi_4 &= \lambda_4^2 + \phi^2\vartheta_2, \\ \theta_4\eta_4 &= \lambda_4'^2 + \phi'^2\vartheta_2, \\ -\lambda_4'\phi &= \lambda_4\phi' + \theta_4\phi'', \end{aligned}$$

so that χ_4 and η_4 may replace χ_4 and λ_4; and

$$\begin{aligned} \xi_4\xi_4' &= \nu_4^2 + \phi^2\vartheta_2'', \\ \psi_4\eta_4' &= \mu_4'^2 + \phi'^2\vartheta_2', \\ -\mu_4'\phi &= \mu_4\phi' + \psi_4\phi'', \end{aligned}$$

so that ξ_4' and η_4' may replace ν_4 and μ_4. Hence *all the simultaneous solutions can be expressed in terms of the algebraically complete set of fifteen constituted by* $\theta_0, \theta_0', \theta_0''$; $\vartheta_2, \vartheta_2', \vartheta_2''$; ϕ, ϕ', ϕ''; χ_4, η_4; ψ_4, η_4'; ξ_4, ξ_4', *a symmetrical set.*

The foregoing equations used for the modification of the system are selected from the following aggregate:

$$\left.\begin{aligned} \lambda_4'\phi + \lambda_4\phi' + \theta_4\phi'' = 0 \\ \mu_4'\phi + \mu_4\phi' + \psi_4\phi'' = 0 \\ \nu_4'\phi + \nu_4\phi' + \xi_4\phi'' = 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} \lambda_4''\phi + \chi_4\phi' + \lambda_4\phi'' = 0 \\ \mu_4''\phi + \theta_4'\phi' + \mu_4\phi'' = 0 \\ \nu_4''\phi + \xi_4'\phi' + \nu_4\phi'' = 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} \theta_4\chi_4 &= \lambda_4^2 + \phi^2\vartheta_2 \\ \psi_4\theta_4' &= \mu_4^2 + \phi^2\vartheta_2' \\ \xi_4\xi_4' &= \nu_4^2 + \phi^2\vartheta_2'' \end{aligned}\right\}, \quad \left.\begin{aligned} \theta_4\eta_4 &= \lambda_4'^2 + \phi'^2\vartheta_2 \\ \psi_2\eta_4' &= \mu_4'^2 + \phi'^2\vartheta_2' \\ \xi_4\theta_4'' &= \nu_4'^2 + \phi'^2\vartheta_2'' \end{aligned}\right\}, \quad \left.\begin{aligned} \chi_4\eta_4 &= \lambda_4''^2 + \phi''^2\vartheta_2 \\ \theta_4'\eta_4' &= \mu_4''^2 + \phi''^2\vartheta_2' \\ \xi_4'\theta_4'' &= \nu_4''^2 + \phi''^2\vartheta_2'' \end{aligned}\right\}.$$

57. As in the system of simultaneous concomitants for two quadratics, there are other solutions of the two characteristic equations (and so other concomitants) simple in form and useful because subsidiary to the expression of concomitants. The most important of these are:

* In the case of a system of n ternary quadratics, it is easy to see (1) that the number of equations in each of the n complete systems formed as above is n^2+2n-1, and (2) that all the simultaneous concomitants are expressible in terms of $6n-3$ concomitants properly chosen.

$$\left.\begin{aligned} f_{12} &= a_2c_0' + a_2'c_0 - 2b_1b_1' \\ f_{23} &= a_2'c_0'' + a_2''c_0' - 2b_1'b_1'' \\ f_{13} &= a_2c_0'' + a_2''c_0 - 2b_1b_1'' \end{aligned}\right\};$$

$$\left.\begin{aligned} g &= (A, B, C \between a_1, -b_0)^2 \\ g' &= (A, B, C \between a_1', -b_0')^2 \\ g'' &= (A, B, C \between a_1'', -b_0'')^2 \end{aligned}\right\}, \quad \left.\begin{aligned} g_{12} &= (A, B, C \between a_1, -b_0 \between a_1', -b_0') \\ g_{13} &= (A, B, C \between a_1, -b_0 \between a_1'', -b_0'') \\ g_{23} &= (A, B, C \between a_1', -b_0' \between a_1'', -b_0'') \end{aligned}\right\},$$

$$\left.\begin{aligned} j &= (A', B', C' \between a_1, -b_0)^2 \\ j' &= (A', B', C' \between a_1', -b_0')^2 \\ j'' &= (A', B', C' \between a_1'', -b_0'')^2 \end{aligned}\right\}, \quad \left.\begin{aligned} j_{12} &= (A', B', C' \between a_1, -b_0 \between a_1', -b_0') \\ j_{13} &= (A', B', C' \between a_1, -b_0 \between a_1'', -b_0'') \\ j_{23} &= (A', B', C' \between a_1', -b_0' \between a_1'', -b_0'') \end{aligned}\right\},$$

$$\left.\begin{aligned} e &= (A'', B'', C'' \between a_1, -b_0)^2 \\ e' &= (A'', B'', C'' \between a_1', -b_0')^2 \\ e'' &= (A'', B'', C'' \between a_1'', -b_0'')^2 \end{aligned}\right\}, \quad \left.\begin{aligned} e_{12} &= (A, B, C \between a_1, -b_0 \between a_1', -b_0') \\ e_{13} &= (A, B, C \between a_1, -b_0 \between a_1'', -b_0'') \\ e_{23} &= (A, B, C \between a_1', -b_0' \between a_1'', -b_0'') \end{aligned}\right\};$$

where

$$\begin{aligned} \tfrac{1}{2}A &= b_1'c_0 - b_1c_0', & B &= a_2'c_0 - a_2c_0', & \tfrac{1}{2}C &= a_2'b_1 - a_2b_1', \\ \tfrac{1}{2}A' &= b_1c_0'' - b_1''c_0, & B' &= a_2c_0'' - a_2''c_0, & \tfrac{1}{2}C' &= a_2b_1'' - a_2''b_1, \\ \tfrac{1}{2}A'' &= b_1''c_0' - b_1'c_0'', & B'' &= a_2''c_0' - a_2'c_0'', & \tfrac{1}{2}C'' &= a_2''b_1' - a_2'b_1''. \end{aligned}$$

And the equations which express these functions in terms of the system of §56 are of the forms

$$\left.\begin{aligned} \tfrac{1}{2}g\phi &= \lambda_4\psi_4 - \mu_4\theta_4 \\ \tfrac{1}{2}g'\phi &= \mu_4\chi_4 - \lambda_4\theta_4' \\ \tfrac{1}{2}g''\phi'' &= \mu_4''\eta_4 - \lambda_4''\eta_4' \end{aligned}\right\}, \quad \left.\begin{aligned} g_{12}\psi_4 &= \mu_4 g + (\psi_4 f_{12} - 2\theta_4\vartheta_2')\,\phi \\ g_{13}\psi_4 &= \mu_4' g + (2\theta_4\vartheta_2' - \psi_4 f_{12})\,\phi' \\ g_{23}\theta_4' &= \nu_4'' g' + (\theta_4' f_{12} - 2\chi_4\vartheta_2')\,\phi'' \end{aligned}\right\}$$

with two similar sets; and

$$\left.\begin{aligned} \tfrac{1}{4}g^2 &= -\vartheta_2'\theta_4^2 + f_{12}\theta_4\psi_4 - \vartheta_2\psi_4^2 \\ \tfrac{1}{4}j^2 &= -\vartheta_2\xi_4^2 + f_{13}\theta_4\xi_4 - \vartheta_2''\theta_4^2 \\ \tfrac{1}{4}e^2 &= -\vartheta_2''\psi_4^2 + f_{23}\psi_4\xi_4 - \vartheta_2'\xi_4^2 \end{aligned}\right\}$$

being one of three sets.

58. The orders in the x-variables and the classes in the u-variables are as follow, being most easily obtained from the symbolical forms:

ORDER.	CLASS.	LEADING COEFFICIENT.
0	2	$\vartheta_2,\ \vartheta_2',\ \vartheta_2''$; $f_{12},\ f_{23},\ f_{13}$.
2	0	$\theta_0,\ \theta_0',\ \theta_0''$.
2	1	$\phi,\ \phi',\ \phi''$.
2	2	$\left.\begin{matrix}\theta_4, & \chi_4, & \eta_4\\ \lambda_4, & \lambda_4', & \lambda_4''\end{matrix}\right\};\ \left.\begin{matrix}\psi_4, & \theta_4', & \eta_4'\\ \mu_4, & \mu_4', & \mu_4''\end{matrix}\right\};\ \left.\begin{matrix}\xi_4, & \xi_4', & \theta_4''\\ \nu_4, & \nu_4', & \nu_4''\end{matrix}\right\}.$
2	3	$\left.\begin{matrix}g, & g', & g''\\ g_{12}, & g_{23}, & g_{13}\end{matrix}\right\};\ \left.\begin{matrix}j, & j', & j''\\ j_{12}, & j_{23}, & j_{13}\end{matrix}\right\};\ \left.\begin{matrix}e, & e', & e''\\ e_{12}, & e_{23}, & e_{13}\end{matrix}\right\}.$

which is to be read: that the concomitant in ϑ_2 as its leading coefficient is of order 0 and class 2, so that its first term is $\vartheta_2 u_1^2$, and so on.

All the simultaneous concomitants can be expressed in terms of the fifteen, which constitute the symmetrical set given by

$$\left.\begin{aligned}U\ &=\theta_0 x_1^2+\dots\\ U'\ &=\theta_0' x_1^2+\dots\\ U''&=\theta_0'' x_1^2+\dots\end{aligned}\right\},\quad \left.\begin{aligned}\Theta_2\ &=\vartheta_2 u_1^2+\dots\\ \Theta_2'\ &=\vartheta_2' u_1^2+\dots\\ \Theta_2''&=\vartheta_2'' u_1^2+\dots\end{aligned}\right\},\quad \left.\begin{aligned}\Phi\ &=\phi x_1^2 u_1+\dots\\ \Phi'\ &=\phi' x_1^2 u_1+\dots\\ \Phi''&=\phi'' x_1^2 u_1+\dots\end{aligned}\right\},$$

$$\left.\begin{aligned}\mathrm{X}_4&=\chi_4 x_1^2 u_1^2+\dots\\ H_4&=\eta_4 x_1^2 u_1^2+\dots\end{aligned}\right\},\quad \left.\begin{aligned}\Psi_4&=\psi_4 x_1^2 u_1^2+\dots\\ H_4'&=\eta_4' x_1^2 u_1^2+\dots\end{aligned}\right\},\quad \left.\begin{aligned}\Xi_4&=\xi_4 x_1^2 u_1^2+\dots\\ \Xi_4'&=\xi_4' x_1^2 u_1^2+\dots\end{aligned}\right\};$$

the symbolical expressions for which are

$$U=\alpha_x^2,\qquad U'=\alpha_x'^2,\qquad U''=\alpha_x''^2;$$
$$\Theta_2=\tfrac{1}{2}(\alpha\beta u)^2,\qquad \Theta_2'=\tfrac{1}{2}(\alpha'\beta' u)^2,\qquad \Theta_2''=\tfrac{1}{2}(\alpha''\beta'' u)^2;$$
$$\Phi=\alpha_x\alpha_x'(\alpha'\alpha u),\qquad \Phi'=\alpha_x''\alpha_x(\alpha\alpha'' u),\qquad \Phi''=\alpha_x'\alpha_x''\,\alpha''\alpha' u);$$
$$\mathrm{X}_4=\beta_x'\gamma_x'(\alpha\beta' u)(\alpha\gamma' u),\qquad H_4=\beta_x''\gamma_x''(\alpha\beta'' u)(\alpha\gamma'' u);$$
$$\Psi_4=\beta_x\gamma_x(\alpha'\beta u)(\alpha'\gamma u),\qquad H_4'=\beta_x''\gamma_x''(\alpha'\beta'' u)(\alpha'\gamma'' u);$$
$$\Xi_4=\beta_x\gamma_x(\alpha''\beta u)(\alpha''\gamma u),\qquad \Xi_4'=\beta_x'\gamma_x'(\alpha''\beta' u)(\alpha''\gamma' u).$$

And for the purposes of expression, the remainder of the set of concomitants determined by the preceding table will be useful.

Thus the Jacobian (Salmon's Conic Sections, §388) is

$$-\frac{1}{u_x}(U\Phi'' + U'\Phi' + U''\Phi);$$

the involutant (ib. §388a) is

$$\frac{1}{8}\frac{GJE}{\Theta_4\Psi_4\Xi_4} - \frac{1}{2}\left(\frac{\Theta_2 E}{\Theta_4} + \frac{\Theta_2' J}{\Psi_4} + \frac{\Theta_2'' G}{\Xi_4}\right);$$

of the ten simultaneous invariants, which are asyzygetic, nine are given by equations similar to those which give the four asyzygetic invariants of two quadratics in §52; and the tenth, being

$$\Sigma a_0(a_2'c_0'' + a_2''c_0' - 2b_1'b_1'') - 2\Sigma\{c_0a_1'a_1'' - b_1(a_1'b_0'' + a_1''b_0') + a_2b_0'b_0''\},$$

is equal to

$$\frac{1}{u_x^2}\{UF_{23} + U'F_{13} + U''F_{12} - 2\Lambda_4'' - 2M_4'' - 2N_4''\}.$$

Similarly for other examples.

Some investigations dealing with a system of three quadratics are given by Cayley and Hermite in the 57th volume of Crelle's Journal, and by Gundelfinger in the 80th volume.

(To be continued.)

Second Memoir on a New Theory of Symmetric Functions.

By Captain P. A. MacMahon, R. A.

In my first memoir on this subject (Vol. XI, No. 1) I introduced the notion of the "separation" of a partition, but restricted myself to the discussion of rational integral symmetric functions.

In the present memoir I am engaged with functions which are not necessarily integral, but require partitions, with positive, zero, and negative parts for their symbolical expression.

The chief results which I obtain are

(i). A simple proof of a generalized Vandermonde-Waring power law which presents itself in the guise of an invariantive property of a transcendental transformation.

(ii). The law of "Groups of Separations."

(iii). The fundamental law of algebraic reciprocity; the proof here given being purely arithmetical.

(iv). The fundamental law of algebraic expressibility which asserts that certain indicated symmetric functions can be exhibited as linear functions of the separations of any given partition.

(v). The existence is established of a pair of symmetrical tables in association with every partition into positive, zero, and negative parts, of every number positive, zero, or negative.

The results (iv) and (v) are immediate deductions from (iii), which I believe to be a theorem of great importance and a natural origin of research in symmetrical algebra.

Attention may be drawn to the free introduction of the zero part into the partitions; this forms a connecting link between arithmetic and algebra, and

enables us to pass in a novel and natural manner from theorems of quantity to theorems of number. An illustration of this may be found at the conclusion of this memoir, where I have given symmetrical tables of binomial coefficients. By employing zero parts, any algebraic function of one quantity may be expressed by means of partitions, and further, every unsymmetrical algebraic function of the quantity x is expressible as a symmetric function of any arbitrary quantities x in number; this is in fact equivalent to the development of $\phi(x)$, a given rational and integral algebraic function of x, in a series of factorials, but it is interesting as showing that all algebra is in reality included in the algebra of symmetric functions; for this reason I think the theorems here given are entitled to rank as theorems in general algebra, and should not be regarded as appertaining exclusively to symmetrical algebra.

In one or two succeeding memoirs I hope to be permitted to further develop the theory of the $X-x$ transformation which possesses many properties of great elegance, and to exhibit, with some approach to completeness, the theory of the allied differential operations, a large and important part of the subject upon which I have not entered in these two memoirs, although I have it by me in manuscript.

Readers should consult "Symmetric Functions and the Theory of Distributions," Lond. Math. Soc., Vol. XIX, p. 220, and "Théorie des Formes Binaires," by Faà de Bruno.

SECTION 1.

1. The theory of symmetric functions is a part of the general theory of permutations, combinations and distributions. Formulae in the former are merely elegant analytical expressions of propositions in the latter theory; this fact I have dwelt upon at some length in a paper, "Symmetric Functions and the Theory of Distributions," Proceedings of the London Mathematical Society, Vol. XIX, p. 220 et seq.

As an illustration, I give the interpretations of two well known theorems in symmetric functions and refer readers to the paper above quoted for the necessary explanations and elucidations.

2. If

$$(1-a_1x+a_2x^2-a_3x^3+\ldots\ldots)^{-1}=1+h_1x+h_2x^2+h_3x^3+\ldots\ldots,$$

then a_m and h_m are designated respectively "the elementary symmetric function of weight m," and "the homogeneous product sum of weight m" of the quantities

$$\alpha, \beta, \gamma, \delta, \ldots,$$

where

$$1 - a_1x + a_2x^2 - a_3x^3 + \ldots = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)\ldots$$

We have the well known theorems

$$(-)^m a_m = \sum \frac{(-)^{\Sigma\lambda}(\Sigma\lambda)!}{\lambda_1!\,\lambda_2!\,\lambda_3!\ldots} h_1^{\lambda_1} h_2^{\lambda_2} h_3^{\lambda_3}\ldots, \tag{i}$$

the summation being controlled by the relation $\Sigma s\lambda_s = m$ and

$$(-)^m h_m = \sum \frac{(-)^{\Sigma\lambda}(\Sigma\lambda)!}{\lambda_1!\,\lambda_2!\,\lambda_3!\ldots} a_1^{\lambda_1} a_2^{\lambda_2} a_3^{\lambda_3}\ldots, \tag{ii}$$

with, as before, the relation $\Sigma s\lambda_s = m$.

3. It will be observed that (ii) is derivable from (i) by the interchange of a and h.

4. These formulae give rise respectively to

Theorem I. "Considering n objects of any species whatever, the number of distinct ways of distributing them into an even number of different parcels is precisely equal to the number of distributions into an uneven number of different parcels, except when the objects are all of different species; in this case, the former number is in excess or in defect of the latter number by unity, according as the number of objects is even or uneven."

5. *Theorem* II. "Considering n objects of any species whatever with the restriction that no parcel may contain two objects of the same species, the number of distributions into an even number of different parcels is in excess or in defect by unity of the number of distributions into an uneven number of different parcels according as n, the number of objects, is even or uneven."

6. In these theorems it is to be understood that the phrase "of any species whatever" means that the objects are not restricted to be all of the same kind or to be all of different kinds, but may be of any kinds whatever; the phrase "different parcels" means that no two parcels are of the same description.

7. As an example of the first theorem, suppose there are four objects, say three pears and an apple, we have the distributions:

Four parcels.	Three parcels.	Two parcels.	One parcel.
p, p, p, a	pp, p, a	pp, pa	$pppa$
p, p, a, p	pp, a, p	pa, pp	
p, a, p, p	p, pp, a	ppp, a	
a, p, p, p	a, pp, p	a, ppp	
	p, a, pp	ppa, p	
	a, p, pp	p, ppa	
	pa, p, p		
	p, pa, p		
	p, p, pa		
No. = 4	9	6	1

and $$4 + 6 = 9 + 1,$$
as stated by the theorem.

8. Again take three different objects, say a pear, an apple, and an orange; the distributions are

Three parcels.	Two parcels.	One parcel.
p, a, o	pa, o	pao
p, o, a	o, pa	
a, p, o	ao, p	
a, o, p	p, ao	
o, p, a	op, a	
o, a, p	a, op	
No. = 6	6	1

and $$6 + 1 - 6 = 1,$$
as stated by the theorem.

9. As an example of the second theorem, take two pears and two apples, and remember that now no two similar objects can appear in the same parcel; we have thus

Four parcels.	Three parcels.	Two parcels.	One parcel.
p, p, a, a	pa, p, a	pa, pa	no way.
p, a, p, a	pa, a, p		
a, p, p, a	p, pa, a		
p, a, a, p	a, pa, p		
a, p, a, p	p, a, pa		
a, a, p, p	a, p, pa		
No. = 6	6	1	0

and $$6 + 1 - (6 + 0) = 1,$$
as should be the case.

Section 2.

The Vandermonde-Waring Law.

10. Referring readers to the "Definitions" given on page 2 of my former memoir, I pass on to a further consideration of the separation theorem given on page 19 (loc. cit.), viz.

$$(-)^{l+m+\dots}\frac{(l+m+\dots-1)!}{l!\,m!\dots}S(\lambda^l\mu^m\dots)$$
$$=\sum(-)^{j_1+j_2+\dots}\frac{(j_1+j_2+\dots-1)!}{j_1!\,j_2!\dots}(J_1)^{j_1}(J_2)^{j_2}\dots,$$

where $S(\lambda^l\mu^m\dots)$ denotes the sum of the n^{th} powers of the quantities expressed by means of separations of the partition $(\lambda^l\mu^m\dots)$ of the number n; $(J_1)^{j_1}(J_2)^{j_2}\dots$ is any one of these separations and the summation is in regard to all the separations.

11. I established this theorem in the Proceedings of the London Mathematical Society, Vol. XIX, p. 247 et seq., but having recently obtained a far simpler proof, I give it here as a preparation for a far more general result which will be established subsequently.

12. Write

$$\begin{aligned}
X_1 &= (1)\,x_1,\\
X_2 &= (2)\,x_2+(1^2)\,x_1^2,\\
X_3 &= (3)\,x_3+(21)\,x_2x_1+(1^3)\,x_1^3,\\
X_4 &= (4)\,x_4+(31)\,x_3x_1+(2^2)\,x_2^2+(21^2)\,x_2x_1^2+(1^4)\,x_1^4,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
X_m &= \Sigma\,(m_1m_2m_3\dots)\,x_{m_1}x_{m_2}x_{m_3}\dots,
\end{aligned}$$

the summation having reference to every partition $(m_1m_2m_3\dots)$ of the number m.

13. We may regard the quantities $X_1, X_2, X_3, \dots$ as transformed into the quantities $x_1, x_2, x_3, \dots$ by means of these relations, and we may enquire whether there exists a system of invariants of this transformation; whether in fact we can form a system of relations between $X_1, X_2, X_3, \dots$ which, to symmetric function multipliers *près*, are equal to the like functions of $x_1, x_2, x_3, \dots$. A complete system of such invariants does exist, and they are of fundamental importance.

In the first place, X_1 is such an invariant; the complete system is found in the following manner:

14. I suppose that the symmetric functions on the dexter of the above relations refer to quantities

$$\alpha, \beta, \gamma, \ldots,$$

which I further consider to be infinite in number.

15. I observe that the expression

$$1 + X_1 + X_2 + X_3 + \ldots$$

may be broken up into factors of the form

$$1 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 + \ldots,$$

so that there is the identity

$$1 + X_1 + X_2 + X_3 + \ldots = \Pi_\alpha (1 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 + \ldots),$$

a factor appearing for each of the quantities

$$\alpha, \beta, \gamma, \ldots;$$

this relation indeed, from another point of view, serves to define the quantities $X_1, X_2, X_3, \ldots$ in a concise manner and *a posteriori* one is directly convinced of its truth.

16. It is convenient to introduce an arbitrary quantity μ and to write

$$1 + \mu X_1 + \mu^2 X_2 + \mu^3 X_3 + \ldots = \Pi_\alpha (1 + \mu\alpha x_1 + \mu^2\alpha^2 x_2 + \mu^3\alpha^3 x_3 + \ldots)$$

Taking logarithms, we find

$$\log(1 + \mu X_1 + \mu^2 X_2 + \mu^3 X_3 + \ldots) = \sum_\alpha \log(1 + \mu\alpha x_1 + \mu_2\alpha^2 x_2 + \mu^3\alpha^3 x_3 + \ldots);$$

the left-hand side of this identity is, when expanded,

$$\mu X_1 + \mu^2\left(X_2 - \frac{1}{2} X_1^2\right) + \mu^3\left(X_3 - X_2 X_1 + \frac{1}{3} X_1^3\right) + \ldots,$$

the general term being

$$\mu^l \sum (-)^{l_1 + l_2 + \ldots + 1} \frac{(l_1 + l_2 + \ldots - 1)!}{l_1!\, l_2! \ldots} X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2} \ldots,$$

where $$l = \sum l\lambda;$$

whereas the right-hand side has the general term

$$\left\{\sum \alpha^l\right\}\mu^l \sum (-)^{l_1+l_2+\ldots+1} \frac{(l_1+l_2+\ldots-1)!}{l_1!\, l_2!\ldots} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots$$

17. Hence, equating coefficients of like powers of μ, we have a system of invariants shown by the relations

$$X_1 = (1)x_1,$$

$$X_2 - \frac{X_1^2}{2} = (2)\left\{x_2 - \frac{x_1^2}{2}\right\},$$

$$X_3 - X_2X_1 + \frac{X_1^3}{3} = (3)\left\{x_3 - x_2x_1 + \frac{x_1^3}{3}\right\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sum (-)^{l_1+l_2+\ldots-1} \frac{(l_1+l_2+\ldots-1)!}{l_1!\, l_2!\ldots} X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2}\ldots$$
$$= (l)\sum (-)^{l_1+l_2+\ldots-1} \frac{(l_1+l_2+\ldots-1)!}{l_1!\, l_2!\ldots} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots$$

18. If we now multiply out the left-hand side in order to find the cofactor therein of

$$x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots$$

we see that the cofactor consists of products of symmetric functions, and that each product is necessarily a separation of the symmetric function

$$(\lambda_1^{l_1}\lambda_2^{l_2}\ldots).$$

Moreover, the coefficient of

$$x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots$$

in the product $X_{\mu_1}^{m_1} X_{\mu_2}^{m_2}\ldots$ is (vide first memoir loc. cit., p. 9)

$$\sum \frac{m_1!\, m_2!\ldots}{j_1!\, j_2!\ldots} (J_1)^{j_1}(J_2)^{j_2}\ldots,$$

wherein $(J_1)^{j_1}(J_2)^{j_2}\ldots$ is any separation of $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)$ of specification $(\mu_1^{m_1}\mu_2^{m_2}\ldots)$.*
Hence

$$(-)^{m_1+m_2+\ldots-1} \frac{(m_1+m_2+\ldots-1)!}{m_1!\, m_2!\ldots} X_{\mu_1}^{m_1} X_{\mu_2}^{m_2}\ldots$$
$$= \ldots + (-)^{m_1+m_2+\ldots-1} \frac{(m_1+m_2+\ldots-1)!}{m_1!\, m_2!\ldots} m_1!\, m_1!\ldots \left\{\sum \frac{(J_1)^{j_1}(J_2)^{j_2}\ldots}{j_1!\, j_2!\ldots}\right\} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots + \ldots$$
$$= \ldots + \left\{\sum (-)^{j_1+j_2+\ldots-1} \frac{(j_1+j_2+\ldots-1)!}{j_1!\, j_2!\ldots} (J_1)^{j_1}(J_2)^{j_2}\ldots\right\} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots + \ldots,$$

* At Professor Cayley's suggestion, I abandon the expression "species partition" in favor of "specification," which is a far more appropriate word.

since $$m_1 + m_2 + \ldots\ldots = j_1 + j_2 + \ldots\ldots$$
Therefore

$$\sum (-)^{l_1+l_2+\ldots\ldots-1} \frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\, l_2!\ldots\ldots} X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2}\ldots\ldots$$
$$= \sum\sum (-)^{j_1+j_2+\ldots\ldots-1} \frac{(j_1+j_2+\ldots\ldots-1)!}{j_1!\, j_2!\ldots\ldots} (J_1)^{j_1}(J_2)^{j_2}\ldots\ldots x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots\ldots,$$

wherein $(J_1)^{j_1}(J_2)^{j_2}\ldots\ldots$ is any separation of $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$.

Hence, substituting

$$\sum\sum (-)^{j_1+j_2+\ldots\ldots-1} \frac{(j_1+j_2+\ldots\ldots-1)!}{j_1!\, j_2!\ldots\ldots} (J_1)^{j_1}(J_2)^{j_2}\ldots\ldots x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots\ldots$$
$$= (l) \sum (-)^{l_1+l_2+\ldots\ldots-1} \frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\, l_2!\ldots\ldots} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots\ldots,$$

and equating coefficients of $x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots\ldots$ we obtain

$$(-)^{l_1+l_2+\ldots\ldots} \frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\, l_2!\ldots} (l)$$
$$= \sum (-)^{j_1+j_2+\ldots\ldots-1} \frac{(j_1+j_2+\ldots\ldots-1)!}{j_1!\, j_2!\ldots\ldots} (J_1)^{j_1}(J_2)^{j_2}\ldots\ldots,$$

which is the theorem to be proved.

19. It will be observed that the theorem arises at once from the invariant property exhibited by the formula

$$\sum (-)^{l_1+l_2+\ldots\ldots} \frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\, l_2!\ldots\ldots} X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2}\ldots\ldots$$
$$= (l) \sum (-)^{l_1+l_2+\ldots\ldots} \frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\, l_2!\ldots\ldots} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots\ldots,$$

an application of the multinomial theorem in algebra being in reality all that is necessary; the formula in fact establishes the theorem at once for all partitions of all numbers, and is itself a condensed and exceedingly elegant analytical representation of it. It is very interesting to find an extensive proposition like the one under view appearing under the guise of an invariant of an algebraical transformation; I remark it particularly, as I have never met with a case at all similar to it before.

Section 3.

Property of the Coefficients of a Group.

20. On page 28 of my former memoir I defined a "Group" as applied to separations of a partition, and I recall here that the separation $(\lambda^2\mu)(\lambda)(\mu)$ belongs to the group

$$G\{(\lambda^2)(\lambda);\ (\mu)^2\}$$

because (λ^3) and (μ^2) occur in the separations $(\lambda^2)(\lambda)$ and $(\mu)^2$ respectively.

21. To put the group in evidence, it is expedient to substitute for the relations between $X_1, X_2, X_3, \ldots$ and $x_1, x_2, x_3, \ldots$ another set, as follows:

$$\begin{aligned}
Y_1 &= (1)\,y_1,\\
Y_2 &= (2)\,y_2 + (1^2)\,y_{1^2},\\
Y_3 &= (3)\,y_3 + (21)\,y_2y_1 + (1^3)\,y_{1^3},\\
Y_4 &= (4)\,y_4 + (31)\,y_3y_1 + (2^2)\,y_{2^2} + (21^2)\,y_2y_{1^2} + (1^4)\,y_{1^4},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
Y_\mu &= \Sigma\,(\mu_1^{m_1}\mu_2^{m_2}\ldots)\,y_{\mu_1{}^{m_1}}y_{\mu_2{}^{m_2}}\ldots\,;
\end{aligned}$$

we then find

$$\begin{aligned}
&Y_4 - Y_3Y_1 - \frac{1}{2}Y_2^2 + Y_2Y_1^2 - \frac{1}{4}Y_1^4\\
&= (4)\,y_4 + \{(31)-(3)(1)\}y_3y_1 + (2^2)\,y_{2^2} - \frac{1}{2}(2)^2y_2^2 + \{(21^2)-(2)(1^2)\}\,y_2y_{1^2}\\
&\quad + \{(2)(1)^2-(21)(1)\}\,y_2y_1^2 + (1^4)\,y_{1^4} - (1^3)(1)\,y_{1^3}y_1 - \frac{1}{2}(1^2)^2y_{1^2}^2\\
&\quad + (1^2)(1)^2\,y_1^2y_{1^2} - \frac{1}{4}(1)^4y_1^4.
\end{aligned}$$

22. Observe that the cofactor of y_3y_1 is composed of members of $G\{(3);(1)\}$,

" " " $y_2y_{1^2}$ " " " $G\{(2);(1^2)\}$,

" " " $y_2y_1^2$ " " " $G\{(2);(1)^2\}$,

and that these are the only y products which are multiplied by separations of a partition composed of different parts. Generally, in the cofactor of a y product,

$$y^{m_1'}_{\mu_1{}^{m_1}}\,y^{m_2'}_{\mu_2{}^{m_2}}\ldots,$$

the equations must belong to the group

$$G\{(\mu_1^{m_1})^{m_1'};\ (\mu_2^{m_2})^{m_2'};\ldots\}.$$

23. I have before given the theorem that if the symmetric function (l) be expressed by means of separations of any partition of l which does not merely consist of repetitions of a single part, the algebraic sum of the coefficients of the separations of each group is zero. To establish this, it is merely necessary to

prove that, forming in succession

$$
\begin{aligned}
&Y_1,\\
&Y_2-\frac{1}{2}Y_1^2,\\
&Y_3-Y_2Y_1+\frac{1}{3}Y_1^3,\\
&Y_4-Y_3Y_1-\frac{1}{2}Y_2^2+Y_2Y_1^2-\frac{1}{4}Y_1^4,\\
&\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

every product $$y^{m_1'}_{\mu_1 m_1} y^{m_2'}_{\mu_2 m_2}$$

vanishes on putting all the symmetric functions (1), (2), (1^2), (3), (21), (1^3), ... equal to unity, unless $\mu_1=\mu_2=\dots.$

24. For this purpose put

$$
\begin{aligned}
'Y_1&=y_1,\\
'Y_2&=y_2+y_{1^2},\\
'Y_3&=y_3+y_2y_1+y_{1^3},\\
'Y_4&=y_4+y_3y_1+y_{2^2}+y_2y_{1^2}+y_{1^4},\\
&\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

so that

$$
\begin{aligned}
&1+'Y_1+'Y_2+'Y_3+\dots\\
&\quad=(1+y_1+y_{1^2}+y_{1^3}+\dots)(1+y_2+y_{2^2}+y_{2^3}+\dots)(1+y_3+y_{3^2}+y_{3^3}+\dots)\dots
\end{aligned}
$$

and taking logarithms

$$
\begin{aligned}
&\log(1+'Y_1+'Y_2+'Y_3+\dots)=\log(1+y_1+y_{1^2}+y_{1^3}+\dots)\\
&\quad+\log(1+y_2+y_{2^2}+y_{2^3}+\dots)+\log(1+y_3+y_{3^2}+y_{3^3}+\dots)+\dots,
\end{aligned}
$$

and on expansion

$$
\begin{aligned}
&'Y_1+\left('Y_2-\frac{1}{2}\,'Y_1^2\right)+\left('Y_3-'Y_2'Y_1+\frac{1}{3}\,'Y_1^3\right)\\
&\quad+\left('Y_4-'Y_3'Y_1-\frac{1}{2}\,'Y_2^2+'Y_2'Y_1^2-\frac{1}{4}\,'Y_1^4\right)+\dots\\
&=y_1+\left(y_{1^2}-\frac{1}{2}y_1^2\right)+\left(y_{1^3}-y_{1^2}y_1+\frac{1}{3}y_1^3\right)\\
&\quad+\left(y_{1^4}-y_{1^3}y_1-\frac{1}{2}y_{1^2}^2+y_{1^2}y_1^2-\frac{1}{4}y_1^4\right)+\dots\\
&+y_2+\left(y_{2^2}-\frac{1}{2}y_2^2\right)+\left(y_{2^3}-y_{2^2}y_2+\frac{1}{3}y_2^3\right)\\
&\quad+\left(y_{2^4}-y_{2^3}y_2-\frac{1}{2}y_{2^2}^2+y_{2^2}y_2^2-\frac{1}{4}y_2^4\right)+\dots\\
&+y_3+\left(y_{3^2}-\frac{1}{2}y_3^2\right)+\left(y_{3^3}-y_{3^2}y_3+\frac{1}{3}y_3^3\right)\\
&\quad+\left(y_{3^4}-y_{3^3}y_3-\frac{1}{2}y_{3^2}^2+y_{3^2}y_3^2-\frac{1}{4}y_3^4\right)+\dots\\
&\quad+\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

which may be written

$$\sum (-)^{l_1+l_2+\ldots\ldots-1} \frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\, l_2! \ldots\ldots} Y^{l_1}_{\lambda_1} Y^{l_2}_{\lambda_2} \ldots\ldots$$

$$= \sum (-)^{l_1+l_2+\ldots\ldots-1} \frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\, l_2! \ldots\ldots} y^{l_1}_{1\lambda_1} y^{l_2}_{1\lambda_2} \ldots\ldots$$

$$+ \sum (-)^{l_1'+l_2'+\ldots\ldots-1} \frac{(l_1'+l_2'+\ldots\ldots-1)!}{l_1'!\, l_2'! \ldots\ldots} y^{l_1'}_{2\lambda_1} y^{l_2'}_{2\lambda_2} \ldots\ldots$$

$$+ \sum (-)^{l_1''+l_2''+\ldots\ldots-1} \frac{(l_1''+l_2''+\ldots\ldots-1)!}{l_1''!\, l_2''! \ldots\ldots} y^{l_1''}_{3\lambda_1} y^{l_2''}_{3\lambda_2} \ldots\ldots + \ldots\ldots$$

where the right-hand side, visibly, contains only products

$$y^{m_1'}_{\mu_1 m_1} y^{m_2'}_{\mu_2 m_2} \ldots\ldots,$$

in which $\mu_1 = \mu_2 = \ldots\ldots$

25. It is thus established that when we express the symmetric function (l) by means of separations of a partition of the number l, which does not merely consist of repetitions of a single part, the algebraic sum of the coefficients in each group of separations is zero.

This proof seems far preferable to the one given in the former memoir.

SECTION 4.

The Theory of Rational Symmetric Functions.

26. I propose to discuss symmetric functions which are rational, but are freed from the restriction of being integral.

Such an expression is $\displaystyle \sum \frac{\alpha^p \beta^q}{\gamma^r} = \sum \alpha^p \beta^q \gamma^{-r};$

attending merely to the indices, this may be written

$$(pq, -r),$$

in which form it appears as a partition with negative as well as positive parts. As far as I have discovered, Meyer Hirsch was the first who employed partitions with negative parts, but neither he nor any subsequent writer appears to have developed this part of the theory (vide Hirsch's Collection of Examples, Formu-

lae and Calculations on the Literal Calculus and Algebra, translated by Rev. J. A. Ross, London, 1827).

27. As a matter of convenience, I write the partition $(p, q, -r)$ in the form $(pq\bar{r})$, and writing the parts of such a partition in descending order of algebraical magnitude, thus:

$$(pq \dots \bar{r}\bar{s}).$$

28. I call p and s respectively the positive and negative degrees of the partition or of the symmetric function.

29. The sum $p+q+\dots-r-s$ is the weight of the partition or symmetric function, or quâ partitions it may be alluded to as the partible number.

30. Strictly speaking, the partition $(pq \dots \bar{r}\bar{s})$ may be spoken of as an algebraic partition of the partible number, but no confusion need arise in the comprehension of what follows if we speak merely of the partition instead of the algebraic partition.

31. For the sake of continuity, as well as for other weighty reasons which will appear, it is advisable to admit the zero as a possible part in such partitions. The general function to be studied then becomes

$$\sum \alpha^p \beta^q \dots \gamma^0 \delta^0 \dots \varepsilon^{-r} \theta^{-s},$$

which may be written

$$(pq \dots 00 \dots \bar{r}\bar{s}),$$

where $p, q, \dots r, s$ are integers.

32. Repetitions of the same part are as usual denoted by power indices, so that

$$(pp000\bar{r}\bar{r}\bar{r})$$

is written

$$(p^2 0^3 \bar{r}^3).$$

33. Regarding $p_1, p_2, p_3, \dots p_s$ as positive or negative integers excluding zero, we have evidently

$$(p_1 p_2 \dots p_s 0) = n - s . (p_1 p_2 \dots p_s),$$

$$(p_1 p_2 \dots p_s 0^2) = \frac{n-s . n-s-1}{1.2} (p_1 p_2 \dots p_s),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

from which we obtain in succession

$$n(p_1p_2\dots p_s) = s(p_1p_2\dots p_s) + (p_1p_2\dots p_s0),$$
$$n^2(p_1p_2\dots p_s) = s^2(p_1p_2\dots p_s) + \overline{2s+1}.(p_1p_2\dots p_s0) + 2(p_1p_2\dots p_s0^2),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

so that the function $(p_1p_2\dots p_s)$ multiplied by any rational integral algebraical function of n is expressible as a linear function of the expressions

$$(p_1p_2\dots p_s),\quad (p_1p_2\dots p_s0),\quad (p_1p_2\dots p_s0^2),\ \dots,$$

in which the coefficients are independent of n.

34. Hence we are considering symmetric functions which are rational algebraic functions of the n quantities

$$\alpha,\ \beta,\ \gamma,\ \dots$$

and at the same time rational and integral algebraic functions of n.

35. Having in view a comprehensive study of the whole theory, I proceed as in the former case and put

$$\begin{aligned}
&1 + X_0\mu^0 + X_1\mu + X_2\mu^2 + \dots \\
&\qquad + X_{-1}\frac{1}{\mu} + X_{-2}\frac{1}{\mu^2} + \dots \\
&= \begin{pmatrix} 1 + \alpha^0x_0\mu^0 + \alpha x_1\mu + \alpha^2x_2\mu^2 + \dots \\ + \frac{1}{\alpha}x_{-1}\frac{1}{\mu} + \frac{1}{\alpha^2}x_{-2}\frac{1}{\mu^2} + \dots \end{pmatrix} \\
&\times \begin{pmatrix} 1 + \beta^0x_0\mu^0 + \beta x_1\mu + \beta^2x_2\mu^2 + \dots \\ + \frac{1}{\beta}x_{-1}\frac{1}{\mu} + \frac{1}{\beta^2}x_{-2}\frac{1}{\mu^2} + \dots \end{pmatrix} \\
&\times \begin{pmatrix} 1 + \gamma^0x_0\mu^0 + \gamma x_1\mu + \gamma^2x_2\mu^2 + \dots \\ + \frac{1}{\gamma}x_{-1}\frac{1}{\mu} + \frac{1}{\gamma^2}x_{-2}\frac{1}{\mu^2} + \dots \end{pmatrix} \\
&\times \text{etc.} \\
&= \prod_\alpha \begin{pmatrix} 1 + \alpha^0x_0\mu^0 + \alpha x_1\mu + \alpha^2x_2\mu^2 + \dots \\ + \frac{1}{\alpha}x_{-1}\frac{1}{\mu} + \frac{1}{\alpha^2}x_{-2}\frac{1}{\mu^2} + \dots \end{pmatrix}.
\end{aligned}$$

36. On multiplying out the right-hand side of this equation, the cofactor of μ^s (s positive, zero, or negative) is found to contain symmetric functions which are symbolical by all partitions of s into positive, zero and negative integers, and moreover, each of these symmetric functions (infinite in number) is attached to the corresponding x product.

Equating coefficients of like powers of μ, we obtain

$$\begin{aligned}
X_0 = {} & (0)\,x_0 + (1\bar{1})\,x_1x_{-1} + (2\bar{2})\,x_2x_{-2} + (2\bar{1}^2)\,x_2x_{-1}^2 \\
& \qquad + (1^2\bar{2})\,x_1^2x_{-2} + (1^2\bar{1}^2)\,x_1^2x_{-1}^2 + \dots \\
& + (0^2)\,x_0^2 + (10\bar{1})\,x_1x_0x_{-1} + (20\bar{2})\,x_2x_0x_{-2} + (20\bar{1}^2)\,x_2x_0x_{-1}^2 \\
& \qquad + (1^20\bar{2})\,x_1^2x_0x_{-2} + (1^20\bar{1}^2)\,x_1^2x_0x_{-1}^2 + \dots \\
& + (0^3)\,x_0^3 + (10^2\bar{1})\,x_1x_0^2x_{-1} + (20^2\bar{2})\,x_2x_0^2x_{-2} + (20^2\bar{1}^2)\,x_2x_0^2x_{-1}^2 \\
& \qquad + (1^20^2\bar{2})\,x_1^2x_0^2x_{-2} + (1^20^2\bar{1}^2)\,x_1^2x_0^2x_{-1}^2 + \dots \\
& + \dots + \dots + \dots + \dots + \dots
\end{aligned}$$

$$\begin{aligned}
X_1 = {} & (1)\,x_1 + (2\bar{1})\,x_2x_{-1} + (1^2\bar{1})\,x_1^2x_{-1} + \dots \\
& + (10)\,x_1x_0 + (20\bar{1})\,x_2x_0x_{-1} + (1^20\bar{1})\,x_1^2x_0x_{-1} + \dots \\
& + (10^2)\,x_1x_0^2 + (20^2\bar{1})\,x_2x_0^2x_{-1} + (1^20^2\bar{1})\,x_1^2x_0^2x_{-1} + \dots \\
& + \dots + \dots + \dots
\end{aligned}$$

$$\begin{aligned}
X_{-1} = {} & (\bar{1})\,x_{-1} + (1\bar{2})\,x_1x_{-2} + (1\bar{1}^2)\,x_1x_{-1}^2 + \dots \\
& + (0\bar{1})\,x_0x_{-1} + (10\bar{2})\,x_1x_0x_{-2} + (10\bar{1}^2)\,x_1x_0x_{-1}^2 + \dots \\
& + (0^2\bar{1})\,x_0^2x_{-1} + (10^2\bar{2})\,x_1x_0^2x_{-2} + (10^2\bar{1}^2)\,x_1x_0^2x_{-1}^2 + \dots \\
& + \dots + \dots + \dots \\
& \text{etc.}
\end{aligned}$$

and generally in the expression of X_s, s being positive, zero, or negative, the summation is taken for every partition of s into positive, zero, and negative integers.

37. Observe that we may write these relations in the form

$$\begin{aligned}
1 + X_0 & = (1+x_0)^n + (1\bar{1})\,x_1(1+x_0)^{n-2}x_{-1} + (2\bar{2})\,x_2(1+x_0)^{n-2}x_{-2} \\
& \quad + (2\bar{1}^2)\,x_2(1+x_0)^{n-3}x_{-1}^2 + (1^2\bar{2})\,x_1^2(1+x_0)^{n-3}x_{-2} + (1^2\bar{1}^2)\,x_1^2(1+x_0)^{n-4}x_{-1}^2 \\
& \quad + \dots \\
X_1 & = (1)\,x_1(1+x_0)^{n-1} + (2\bar{1})\,x_2(1+x_0)^{n-2}x_{-1} + (1^2\bar{1})\,x_1^2(1+x_0)^{n-3}x_{-1} + \dots, \\
X_{-1} & = (\bar{1})(1+x_0)^{n-1}x_{-1} + (1\bar{2})\,x_1(1+x_0)^{n-2}x_{-2} + (1\bar{1}^2)\,x_1(1+x_0)^{n-3}x_{-1}^2 + \dots, \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

38. And also in the forms

$$
\begin{aligned}
\frac{1+X_0}{(1+x_0)^n} = 1 + (1\overline{1})\frac{x_1x_{-1}}{(1+x_0)^2} + (2\overline{2})\frac{x_2x_{-2}}{(1+x_0)^2} + (2\overline{1}^2)\frac{x_2x_{-1}^2}{(1+x_0)^3} \\
+ (1^2\overline{2})\frac{x_1^2x_{-2}}{(1+x_0)^{n-3}} + (1^2\overline{1}^2)\frac{x_1^2x_{-1}^2}{(1+x_0)^4} + \ldots\ldots,
\end{aligned}
$$

$$
\frac{X_1}{(1+x_0)^n} = (1)\frac{x_1}{1+x_0} + (2\overline{1})\frac{x_2x_{-1}}{(1+x_0)^2} + (1^2\overline{1})\frac{x_1^2x_{-1}}{(1+x_0)^3} + \ldots\ldots,
$$

$$
\frac{X_{-1}}{(1+x_0)^n} = (\overline{1})\frac{x_{-1}}{1+x_0} + (1\overline{2})\frac{x_1x_{-2}}{(1+x_0)^2} + (1\overline{1}^2)\frac{x_1x_{-1}^2}{(1+x_0)^3} + \ldots\ldots
$$

. . . .

39. These relations may be regarded as defining a transformation of

$$
\begin{array}{llll}
X_0,\ X_1,\ X_2,\ \ldots\ldots & \text{into functions of} & x_0,\ x_1,\ x_2,\ \ldots\ldots, \\
\quad X_{-1},\ X_{-2},\ \ldots\ldots & \quad\text{"}\qquad\text{"}\qquad\text{"} & \quad x_{-1},\ x_{-2},\ \ldots\ldots,
\end{array}
$$

and we may seek the invariants of the transformation.

40. Recalling the relation

$$
\begin{aligned}
1 + X_0\mu^0 + X_1\mu + X_2\mu^2 + \ldots\ldots &= \prod_\alpha \{1 + \alpha^0x_0\mu^0 + \alpha x_1\mu + \alpha^2x_2\mu^2 + \ldots\ldots\}, \\
+ X_{-1}\frac{1}{\mu} + X_{-2}\frac{1}{\mu^2} \qquad\qquad & \qquad\qquad + \frac{1}{\alpha}x_{-1}\frac{1}{\mu} + \frac{1}{\alpha^2}x_{-2}\frac{1}{\mu^2},
\end{aligned}
$$

and taking logarithms, we find

$$
\begin{aligned}
\log(1 + X_0\mu^0 + X_1\mu + X_2\mu^2 + \ldots) &= \sum_\alpha \log(1 + \alpha^0x_0\mu^0 + \alpha x_1\mu + \alpha^2x_2\mu^2 + \ldots), \\
+ X_{-1}\frac{1}{\mu} + X_{-2}\frac{1}{\mu^2} \qquad\qquad & \qquad\qquad + \frac{1}{\alpha}x_{-1}\frac{1}{\mu} + \frac{1}{\alpha^2}x_{-2}\frac{1}{\mu^2},
\end{aligned}
$$

which may be written

$$
\begin{aligned}
&\log(1 + X_0) + \log\left\{1 + \frac{X_1}{1+X_0}\mu + \frac{X_2}{1+X_0}\mu^2 + \ldots\ldots\right. \\
&\qquad\qquad\qquad\qquad \left. + \frac{X_{-1}}{1+X_0}\frac{1}{\mu} + \frac{X_{-2}}{1+X_0}\frac{1}{\mu^2} + \ldots\ldots\right\} \\
&= \sum_\alpha \left[\log(1 + x_0) + \log\left\{1 + \alpha\frac{x_1}{1+x_0}\mu + \alpha^2\frac{x_2}{1+x_0}\mu^2 + \ldots\ldots\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. + \frac{1}{\alpha}\frac{x_{-1}}{1+x_0}\frac{1}{\mu} + \frac{1}{\alpha^2}\frac{x_{-2}}{1+x_0}\frac{1}{\mu^2} + \ldots\ldots\right\}\right]
\end{aligned}
$$

41. We may now expand each side by the multinomial theorem and equate coefficients of like powers of μ.

Taking first the zero power of μ, we have

$$
\begin{aligned}
&\log(1+X_0) - \frac{X_1X_{-1}}{(1+X_0)^2} \\
&\qquad - \frac{X_2X_{-2}}{(1+X_0)^2} + \frac{X_1^2X_{-2}}{(1+X_0)^3} \\
&\qquad + \frac{X_2X_{-1}^2}{(1+X_0)^3} - \frac{3}{2}\,\frac{X_1^2X_{-1}^2}{(1+X_0)^4} \\
&\qquad - \frac{X_3X_{-3}}{(1+X_0)^2} + 2\,\frac{X_2X_1X_{-3}}{(1+X_0)^3} - \frac{X_1^3X_{-3}}{(1+X_0)^4} \\
&\qquad + 2\,\frac{X_3X_{-2}X_{-1}}{(1+X_0)^3} - 6\,\frac{X_2X_1X_{-2}X_{-1}}{(1+X_0)^4} + 4\,\frac{X_1^3X_{-2}X_{-1}}{(1+X_0)^5} \\
&\qquad - \frac{X_3X_{-1}^3}{(1+X_0)^4} + 4\,\frac{X_2X_1X_{-1}^3}{(1+X_0)^5} - \frac{10}{3}\,\frac{X_1^3X_{-1}^3}{(1+X_0)^6} \\
&\qquad + \dots \\
= (0)\Big[&\log(1+x_0) - \frac{x_1x_{-1}}{(1+x_0)^2} \\
&\qquad - \frac{x_2x_{-2}}{(1+x_0)^2} + \frac{x_1^2x_{-2}}{(1+x_0)^3} \\
&\qquad + \frac{x_2x_{-1}^2}{(1+x_0)^3} - \frac{3}{2}\,\frac{x_1^2x_{-1}^2}{(1+x_0)^4} \\
&\qquad - \frac{x_3x_{-3}}{(1+x_0)^2} + 2\,\frac{x_2x_1x_{-3}}{(1+x_0)^3} - \frac{x_1^3x_{-3}}{(1+x_0)^4} \\
&\qquad + 2\,\frac{x_3x_{-2}x_{-1}}{(1+x_c)^3} + 6\,\frac{x_2x_1x_{-2}x_{-1}}{(1+x_0)^4} + 4\,\frac{x_1^3x_{-2}x_{-1}}{(1+x_0)^5} \\
&\qquad - \frac{x_3x_{-1}^3}{(1+x_0)^4} + 4\,\frac{x_2x_1x_{-1}^3}{(1+x_0)^5} - \frac{10}{3}\,\frac{x_1^3x_{-1}^3}{(1+x_0)^6} \\
&\qquad + \dots\dots\dots\dots\dots\dots\Big],
\end{aligned}
$$

from which it appears that the left-hand side of the identity is an invariant of the transformation.

42. Observe that this invariant consists of a logarithmic term, together with an infinite succession of square blocks of terms; each of these blocks possesses row and column symmetry, both as regards the numerical coefficients and as regards the forms of the X products.

43. An invariant is likewise obtained from every other power of μ positive and negative, thus:

$$
\begin{aligned}
&\frac{X_1}{1+X_0}\\
&-\frac{X_2X_{-1}}{(1+X_0)^2}+\frac{X_1^2X_{-1}}{(1+X_0)^3}\\
&-\frac{X_3X_{-2}}{(1+X_0)^2}+2\frac{X_2X_1X_{-2}}{(1+X_0)^3}-\frac{X_1^3X_{-2}}{(1+X_0)^4}\\
&+\frac{X_3X_{-1}^2}{(1+X_0)^3}-3\frac{X_2X_1X_{-1}^2}{(1+X_0)^4}+2\frac{X_1^3X_{-1}^2}{(1+X_0)^5}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
=(1)\Big[&\frac{x_1}{1+x_0}\\
&-\frac{x_2x_{-1}}{(1+x_0)^2}+\frac{x_1^2x_{-1}}{(1+x_0)^3}\\
&-\frac{x_3x_{-2}}{(1+x_0)^2}+2\frac{x_2x_1x_{-2}}{(1+x_0)^3}-\frac{x_1^3x_{-2}}{(1+x_0)^4}\\
&+\frac{x_3x_{-1}^2}{(1+x_0)^3}-3\frac{x_2x_1x_{-1}^2}{(1+x_0)^4}+2\frac{x_1^3x_{-1}^2}{(1+x_0)^5}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{X_{-1}}{1+X_0}\\
&-\frac{X_1X_{-2}}{(1+X_0)^2}+\frac{X_1X_{-1}^2}{(1+X_0)^3}\\
&-\frac{X_2X_{-3}}{(1+X_0)^2}+2\frac{X_2X_{-1}X_{-2}}{(1+X_0)^3}-\frac{X_2X_{-1}^3}{(1+X_0)^4}\\
&+\frac{X_1^2X_{-3}}{(1+X_0)^3}-3\frac{X_1^2X_{-1}X_{-2}}{(1+X_0)^4}+2\frac{X_1^2X_{-1}^3}{(1+X_0)^5}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
=(\bar{1})\Big[&\frac{x_{-1}}{1+x_0}\\
&-\frac{x_1x_{-2}}{(1+x_0)^2}+\frac{x_1x_{-1}^2}{(1+x_0)^3}\\
&-\frac{x_2x_{-3}}{(1+x_0)^2}+2\frac{x_2x_{-1}x_{-2}}{(1+x_0)^3}-\frac{x_2x_{-1}^3}{(1+x_0)^4}\\
&+\frac{x_1^2x_{-3}}{(1+x_0)^3}-3\frac{x_1^2x_{-1}x_{-2}}{(1+x_0)^4}+2\frac{x_1^2x_{-1}^3}{(1+x_0)^5}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big],
\end{aligned}
$$

and so forth.

44. These invariants may be written

$$\log(1+X_0)+\sum(-)^{l_1+l_2+\dots-1}\frac{(l_1+l_2+\dots-1)!}{l_1!\,l_2!\dots}=\left(\frac{X_{\lambda_1}}{1+X_0}\right)^{l_1}\left(\frac{X_{\lambda_2}}{1+X_0}\right)^{l_2}\dots,$$

the summation being for all solutions of the equation

$$l_1\lambda_1+l_2\lambda_2+\dots=0,$$

in positive and negative integers, but excluding zero; and

$$\sum(-)^{l_1+l_2+\dots-1}\frac{(l_1+l_2+\dots-1)!}{l_1!\,l_2!\dots}\left(\frac{X_{\lambda_1}}{1+X_0}\right)^{l_1}\left(\frac{X_{\lambda_2}}{1+X_0}\right)^{l_2}\dots,$$

where the summation is for all integer and non-zero solutions of the equation

$$l_1\lambda_1+l_2\lambda_2+\dots=m,$$

m being any positive or negative but not a zero integer.

45. We may expand the logarithm in the invariant of weight zero, and moreover in all the invariants we may expand the factors

$$\left(\frac{1}{1+X_0}\right)^{l_1},\ \left(\frac{1}{1+X_0}\right)^{l_2},\dots,$$

and we see that we may write the whole system of invariants in the form

$$\sum(-)^{l_1+l_2+\dots-1}\frac{(l_1+l_2+\dots-1)!}{l_1!\,l_2!\dots}X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}\dots,$$

where now $\qquad l_1\lambda_1+l_2\lambda_2+\dots=m;$

m may be any integer, positive, zero or negative, and the summation is in regard to all solutions of the indeterminate equation

$$l_1\lambda_1+l_2\lambda_2+\dots=m,$$

in positive, zero and negative integers.

46. We can now enunciate as follows:

Theorem. If

$$\begin{aligned}1+X_0+X_1&+X_2+\dots=\prod_\alpha(1+\alpha^0x_0+\alpha x_1+\alpha^2x_2+\dots),\\ &+X_{-1}+X_{-2}\qquad\qquad\qquad +\frac{1}{\alpha}x_{-1}+\frac{1}{\alpha^2}x_{-2}\end{aligned}$$

then

$$\sum(-)^{l_1+l_2+\dots-1}\frac{(l_1+l_2+\dots-1)!}{l_1!\,l_2!\dots}X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}\dots$$
$$=(m)\sum(-)^{l_1+l_2+\dots-1}\frac{(l_1+l_2+\dots-1)!}{l_1!\,l_2!\dots}x_{\lambda_1}^{l_1}x_{\lambda_2}^{l_2}\dots,$$

where the summations are for all solutions of the indeterminate equation

$$l_1\lambda_1 + l_2\lambda_2 + \ldots\ldots = m,$$

in positive, zero, and negative integers, and m is any integer, positive, zero, or negative.

47. This invariant property that has just been established is fundamental and of very great importance.

We now proceed as in the previous more simple case, to multiply out the sinister of the identity, in order to find out therein the cofactor of $x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2} \ldots\ldots$; this cofactor is an assemblage of symmetric function products, each of which is symbolized by a separation of the partition $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$, and we obtain the numerical coefficients by application of the ordinary multinomial theorem: the reasoning is the same as in the previous case, and we are thus led to the comprehensive theorem

$$(-)^{l_1+l_2+\ldots\ldots-1}\frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\,l_2!\ldots\ldots}(m)$$
$$=\sum(-)^{j_1+j_2+\ldots\ldots-1}\frac{(j_1+j_2+\ldots\ldots-1)!}{j_1!\,j_2!\ldots\ldots}(J_1)^{j_1}(J_2)^{j_2}\ldots\ldots,$$

or, as this may be written,

48. $$(-)^{l_1+l_2+\ldots\ldots-1}\frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\,l_2!\ldots\ldots}S(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$$
$$=\sum(-)^{j_1+j_2+\ldots\ldots-1}\frac{(j_1+j_2+\ldots\ldots-1)!}{j_1!\,j_2!\ldots\ldots}(J_1)^{j_1}(J_2)^{j_2}\ldots\ldots,$$

wherein $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$ is any partition of $m\,(=\Sigma l\lambda)$ into positive, zero and negative integers; $S(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$ denotes the symmetric function (m) expressed by means of separations of the partition $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$ of the number m; $(J_1)^{j_1}(J_2)^{j_2}\ldots\ldots$ is any separation of the partition

$$(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots);$$

and the summation is in regard to all such separations. Two examples of this theorem are subjoined.

Example I.

49. To express the symmetric function (2) by means of separations of the symmetric function $(210\bar{1})$.

We form two columns, the first consisting of the different separations, and the second involving the coefficients given by the theorem. We thus have

Separable Partition	Coefficient
$(210\bar{1})$	-6

Separations	Coefficients
$(2)(1)(0)(\bar{1})$	-6
$(21)(0)(\bar{1})$	$+2$
$(20)(1)(\bar{1})$	$+2$
$(2\bar{1})(1)(0)$	$+2$
$(10)(2)(\bar{1})$	$+2$
$(1\bar{1})(2)(0)$	$+2$
$(0\bar{1})(2)(1)$	$+2$
$(21)(0\bar{1})$	-1
$(20)(1\bar{1})$	-1
$(2\bar{1})(10)$	-1
$(210)(\bar{1})$	-1
$(21\bar{1})(0)$	-1
$(20\bar{1})(1)$	-1
$(10\bar{1})(2)$	-1
$(210\bar{1})$	$+1.$

Hence

$$\begin{aligned}-6(2) = &-6(2)(1)(0)(\bar{1})\\ &+2\{(21)(0)(\bar{1})+(20)(1)(\bar{1})+(2\bar{1})(1)(0)+(10)(2)(\bar{1})\\ &\qquad\qquad+(1\bar{1})(2)(0)+(0\bar{1})(2)(1)\}\\ &-\{(21)(0\bar{1})+(20)(1\bar{1})+(2\bar{1})(10)\}\\ &-\{(210)(\bar{1})+(21\bar{1})(0)+(20\bar{1})(1)+(10\bar{1})(2)\}\\ &+(210\bar{1}).\end{aligned}$$

50. To verify this identity, observe that

$$\begin{aligned}(0) &= n,\\ (\lambda 0) &= n-1.(\lambda),\\ (\lambda\mu 0) &= n-2.(\lambda\mu),\end{aligned}$$

so that the identity leads to

$$
\begin{aligned}
-6(2) = &- 6n(2)(1)(\bar{1}) \\
&+ 2n\{(21)(\bar{1}) + (2\bar{1})(1) + (1\bar{1})(2)\} \\
&+ 6(n-1)(2)(1)(\bar{1}) \\
&-(n-1)\{(21)(\bar{1}) + (2\bar{1})(1) + (1\bar{1})(2)\} \\
&- n(21\bar{1}) \\
&-(n-2)\{(21)(\bar{1}) + (2\bar{1})(1) + (1\bar{1})(2)\} \\
&+ (n-3)(21\bar{1});
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
+2(2) = &+ 2(2)(1)(\bar{1}) \\
&- \{(21)(\bar{1}) + (2\bar{1})(1) + (1\bar{1})(2)\} \\
&+ (21\bar{1}),
\end{aligned}
$$

a result which is precisely that given by the theorem for the expression of (2) by means of separations of

$$(21\bar{1}).$$

51. Written in the algebraic form, this last result is

$$2\Sigma\alpha^2 = 2\Sigma\alpha^2\Sigma\alpha'\Sigma\alpha^{-1} - \Sigma\alpha^2\beta'\Sigma\alpha^{-1} - \Sigma\alpha^2\beta^{-1}\Sigma\alpha' - \Sigma\alpha'\beta^{-1}\Sigma\alpha^2 + \Sigma\alpha^2\beta'\gamma^{-1}.$$

52. Example II.

To express S_3 by means of separations of $(3^3\bar{3}^2)$. The result arranged by groups is as follows:

$$
\begin{array}{lllllll}
2S_3 = & 2(3)^3(\bar{3})^2 & -(3)^3(\bar{3}^2) & -3(3^2)(3)(\bar{3})^2 & +2(3^2)(3)(\bar{3}^2) & +(3^3)(\bar{3})^2 & -(3^3)(\bar{3}^2) \\
 & & & & & & -(3^3)(\bar{3}^2) \\
 & -3(3)^2(3\bar{3})(\bar{3}) & +(3)^2(3\bar{3}^2) & +2(3^2\bar{3})(3)(\bar{3}) & -(3^2\bar{3}^2)(3) & -(3^3\bar{3})(\bar{3}) & +(3^3\bar{3}^2) \\
 & +(3)(3\bar{3})^2 & & +2(3^2)(3\bar{3})(\bar{3}) & -(3^2)(3\bar{3}^2) & & \\
 & & & -(3^2\bar{3})(3\bar{3}) & & & \\
\hline
 & \pm 3 & \pm 1 & \pm 4 & \pm 2 & \pm 1 & \pm 1 \\
\hline
\end{array}
$$

53. To establish the law that the algebraic sum of the coefficients in each group is constant, we proceed, as in paragraph 24, and put

$$
\begin{aligned}
1 + {}'Y_0 + {}'Y_1 + {}'Y_2 + \ldots \\
= (1 + y_0 + y_{0^2} + y_{0^3} + \ldots)(1 + y_1 + y_{1^2} + y_{1^3} + \ldots)(1 + y_2 + y_{2^2} + y_{2^3} + \ldots)\ldots \\
+ {}'Y_{-1} + {}'Y_{-2} \qquad \times (1 + y_{\bar{1}} + y_{\bar{1}^2} + y_{\bar{1}^3} + \ldots)(1 + y_{\bar{2}} + y_{\bar{2}^2} + y_{\bar{2}^3} + \ldots)\ldots
\end{aligned}
$$

Now, taking logarithms, the demonstration proceeds *pari passu* with the former and simpler case.

Section 5.

The Law of Reciprocity.

54. I pass on to the generalization of the law of reciprocity which was established in the former memoir, p. 3 et seq.

55. The theorem to be proved is:

Theorem. "Writing

$$\begin{aligned} 1 + X_0\mu^0 + X_1\mu \quad + X_2\mu^2 + \ldots &= \prod_\alpha \left(1 + \alpha^0 x_0\mu^0 + \alpha x_1\mu \quad + \alpha^2 x_2\mu^2 + \ldots\right), \\ + X_{-1}\frac{1}{\mu} + X_{-2}\frac{1}{\mu^2} \qquad\qquad & \qquad\qquad + \frac{1}{\alpha}x_{-1}\frac{1}{\mu} + \frac{1}{\alpha^2}x_{-2}\frac{1}{\mu^2}, \end{aligned}$$

where the product extends to each of the n quantities

$$\alpha, \beta, \gamma, \ldots \qquad (n = \infty),$$

and forming and developing the product

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots,$$

we obtain a result

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots = \ldots + \theta\,(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots,$$

θ being the numerical coefficient of the term

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots,$$

in the development of the product

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots;$$

then $\quad X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}X_{\lambda_3}^{l_3}\ldots = \ldots + \theta\,(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots,$

that is to say, the coefficient of the term

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots$$

in the development of the product

$$X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}X_{\lambda_3}^{l_3}\ldots$$

is the same number θ."

56. The proof here presented is, as was the one in the former memoir, purely arithmetical in its nature, and depends upon the consideration of a particular mode of distribution of a given number of objects into the same number

of parcels, no parcel being empty; we have invariably one object in each parcel. The distribution is of a more general character than the one previously considered and includes the latter as a particular case. It will be seen that when once the character of the distribution has been precisely defined and its connection, with the subject treated of, established by a close examination of a particular case, the actual proof is instantaneous; it arises in fact from a single observation which is of such an elementary character that it admits of no dispute.

It is necessary to make some definitions more extended than those given in Proc. Lond. Math. Soc., Vol. XIX, p. 243.

57. Suppose any number of objects, all of the same kind, to be separated into an upper group and a lower group, in such wise that the upper group consists of λ_1 more objects than the lower group; such an assemblage of similar objects, so separated, may be spoken of as "Objects of type (λ_1)"; the actual number of objects is immaterial; so long as the number of objects in the upper group exceeds the number in the lower group by λ_1, the objects are of type (λ_1).

58. The number λ_1 may be positive, zero, or negative.

Ex. gr. Objects of type (0) may be $\frac{a}{a}$ or $\frac{aaa}{aaa}$ or etc.

and objects of type $(\bar{2})$ may be $\frac{\cdot}{aa}$ or $\frac{a}{aaa}$ or $\frac{aaa}{aaaaa}$ or etc.

59. I make a distinction between "Objects of type (λ_1)" and "Objects (λ_1)."

I consider "Objects of *type* (λ_1)" to have reference to objects of any, the same, kind, so that objects

$$\frac{a}{a} \text{ or } \frac{b}{b}, \text{ etc.}$$

are alike of type (0); whereas, when the objects are restricted to be of a certain definite kind a, I speak of "Objects (λ_1)."

60. Again: "Objects of type $(\lambda_1\lambda_2\lambda_3 \ldots .)$" is defined to mean

(i). "Objects of type (λ_1) of one kind.
(ii). "Objects of type (λ_2) of a second kind.
(iii). "Objects of type (λ_3) of a third kind.
. . . . ;

thus "objects of type $(30\bar{1})$" may be such as

$$\begin{matrix} ccc & ddd & ee \\ & ddd & eee \end{matrix} \text{ or } \begin{matrix} eeee & f & g \\ e & f & gg \end{matrix},$$

where the species of object obtaining in each group is not specified; whereas, if it be stated or implied that the objects in the three groups are of given species, say a, b, c respectively, we would speak of "objects $(30\bar{1})$"; then "objects $(30\bar{1})$" might mean an assemblage such as

$$\begin{array}{lll} aaaaa & bb & \\ aa & bb & c \end{array},$$

the excesses of the objects in the upper group over those in the lower groups being respectively $3, 0$ and -1.

The distinction made between "objects of type $(\lambda_1\lambda_2\lambda_3\ldots.)$" and "objects $(\lambda_1\lambda_2\lambda_3\ldots.)$" will be now understood.

61. Observe that "objects (0)" refers to a set of at least two objects, one in each group.

62. If no restriction be placed upon the number of objects, there is an infinite number of assemblages included in the phrase "objects $(\lambda_1\lambda_2\lambda_3\ldots.)$"; by fixing the number of objects we obtain a finite number of assemblages; fixing the number of objects at 8, "objects $(30\bar{1})$" will comprise the three assemblages:

$$\begin{array}{lll} aaa & b & c \\ & b & cc \end{array}; \quad \begin{array}{lll} aaa & bb & \\ & bb & c \end{array}; \quad \begin{array}{lll} aaaa & b & \\ a & b & c \end{array}.$$

63. We have now objects of various kinds, divided into upper and lower groups, and we may have boxes or parcels of various kinds, similarly divided into certain upper and lower groups, to contain these objects in such wise that one parcel of an upper group contains one object of an upper group, and one parcel of a lower group contains one object of a lower group, there being as many parcels as objects.

64. "Parcels of type $(\lambda_1\lambda_2\lambda_3\ldots.)$" and "Parcels $(\lambda_1\lambda_2\lambda_3\ldots.)$" are defined precisely as in the case of objects, capital letters being employed to exhibit them instead of small ones.

65. Thus 9 "Parcels $(10^2\bar{2})$" will comprise the four assemblages of parcels:

$$\begin{array}{llll} A & B & C & D \\ & B & C & DDD \end{array}; \quad \begin{array}{llll} A & BB & C & \\ & BB & C & DD \end{array};$$

$$\begin{array}{llll} A & B & CC & \\ & B & CC & DD \end{array}; \quad \begin{array}{llll} AA & B & C & \\ A & B & C & DD \end{array}.$$

66. Let us now take 8 "objects $(30\bar{1})$," viz. the three assemblages

$$\begin{array}{ccc} aaa & b & c \\ & b & cc \end{array}; \quad \begin{array}{cc} aaa & bb \\ bb & c \end{array}; \quad \begin{array}{ccc} aaaa & b & \\ a & b & c \end{array};$$

and also 8 "parcels $(4\bar{2})$," viz. the two assemblages

$$\begin{array}{cc} AAAAA & \\ A & BB \end{array}; \quad \begin{array}{cc} AAAA & B \\ & BBB \end{array}.$$

We make a distribution of 8 "objects $(30\bar{1})$" into 8 "parcels $(4\bar{2})$" by placing the objects which occur in any one of the assemblages of objects into the parcels which occur in any one of the assemblages of parcels, in such wise that objects of upper and lower groups appear only in parcels of upper and lower groups respectively, and one parcel contains one and only one object.

67. This distribution is practicable because the partitions

$$(30\bar{1}) \text{ and } (4\bar{2})$$

are each of the same weight, viz. 2. In this manner a definite number of distributions is obtained.

Let us place the second assemblage of objects in the first assemblage of parcels: thus, as one case, we have

$$\begin{array}{ll} AAAAA & \\ a\,a\,a\,b\,b & \\ A & BB \\ b & b\,c \end{array}.$$

68. An examination of this distribution shows us that we can separate it into four portions, so that each portion consists of but one kind of parcel and of but one kind of object; the four portions are

$$\left|\begin{array}{c|c|c|c} \text{I} & \text{II} & \text{III} & \text{IV} \\ AAA & AA & & \\ a\,a\,a & b\,b & & \\ & A & B & B \\ & b & b & c \end{array}\right|,$$

wherein portion I contains "objects of type (3)" placed in "parcels of type (3),"
II " " " (1)" " " " " (1),"
III " " " $(\bar{1})$" " " " " $(\bar{1})$,"
IV " " " $(\bar{1})$" " " " " $(\bar{1})$";

this particular case of distribution possesses a property which is indicated by the succession of numbers 3, 1, -1, -1; thus the property may be defined by the partition $(31\bar{1}^2)$ whose weight is 2, which is of necessity the same as that common to the partitions $(30\bar{1})$, $(4\bar{2})$ which define the assemblages of objects and parcels.

69. We may now restrict ourselves to those distributions of assemblages of objects into assemblages of parcels which possess the property defined by the partition $(31\bar{1}^2)$.

70. This partition $(31\bar{1}^2)$ will be spoken of as the "partition of restriction."

71. The whole number of distributions of assemblages of objects $(30\bar{1})$ into assemblages of parcels $(4\bar{2})$, subject to the restriction of partition $(31\bar{1}^2)$, are now given; they are four in number, viz.

$$\left\{\begin{array}{ll} AAAAA & \\ aaabb & \\ A & BB \\ b & bc \end{array}\right. ,$$

$$\left\{\begin{array}{ll} AAAAA & \\ aaaab & \\ A & BB \\ a & bc \end{array}\right. ,$$

$$\left\{\begin{array}{ll} AAAA & B \\ aaab & c \\ & BBB \\ & bcc \end{array}\right. ,$$

$$\left\{\begin{array}{ll} AAAA & B \\ aaab & b \\ & BBB \\ & bbc \end{array}\right. .$$

72. It is to be understood that the distributions now under examination are connected with three partitions of the same number; the partition of the objects, the partition of the parcels, and the partition of restriction.

73. The weight of the partitions may be any integer, positive, zero, or negative.

74. The number of objects may be any whatever, subject merely to a lower limit which is fixed by the partition of restriction; if a positive part λ occur in this partition, λ objects at least are thereby implied; a negative part $\bar{\lambda}$ also implies at least λ objects, whilst each part zero necessitates at least two objects; thus if p be the sum of the positive parts, if there be q zeros, and if r be the sum of the negative parts,

$$p + 2q + r$$

is a lower limit to the number of objects which can be taken, while in general we may take $p + 2q + r + 2m$ objects, where m is zero or any positive integer.

75. For present purposes it is necessary to consider a minimum number of objects as taking part in the distributions; this, as above mentioned, is known as soon as we decide upon the partition of restriction.

In the example already given, 8 objects were taken, but 6 objects may be taken, as is evident from the partition of restriction $(31\bar{1}^2)$. Reducing the number to 6, we find one assemblage of objects

$$\begin{array}{ll} aaa & b \\ & b\ \ c \end{array}$$

and also one assemblage of parcels

$$\begin{array}{ll} AAAA & \\ & BB \end{array}$$

and subject to the restriction, but one distribution, viz.

$$\left\{\begin{array}{ll} AAAA & \\ a\,a\,a\,b & \\ & BB \\ & b\,c \end{array}\right.$$

76. In general, therefore, our distributions are precisely defined by three partitions of the same number, and in every case their number will be perfectly definite.

77. It is now necessary to make a minute examination of a particular case of the general theorem, in order that we may see the bearing of this theory of distribution upon the multiplication of symmetric functions.

78. Since

$$
\begin{aligned}
X_1 = {} & (1)\,x_1 + (10)\,x_1x_0 + (10^2)\,x_1x_0^2 + \dots\dots \\
& + (2\bar{1})\,x_2x_{-1} + (20\bar{1})\,x_2x_0x_{-1} + (20^2\bar{1})\,x_2x_0^2x_{-1} + \dots\dots \\
& + (1^2\bar{1})\,x_1^2x_{-1} + (1^20\bar{1})\,x_1^2x_0x_{-1} + (1^20^2\bar{1})\,x_1^2x_0^2x_{-1} + \dots\dots \\
& + (3\bar{2})\,x_3x_{-2} + (30\bar{2})\,x_3x_0x_{-2} + (30^2\bar{2})\,x_3x_0^2x_{-2} + \dots\dots \\
& + (3\bar{1}^2)\,x_3x_{-1}^2 + (30\bar{1}^2)\,x_3x_0x_{-1}^2 + (30^2\bar{1}^2)\,x_3x_0^2x_{-1}^2 + \dots\dots \\
& + (21\bar{2})\,x_2x_1x_{-2} + (210\bar{2})\,x_2x_1x_0x_{-2} + (210^2\bar{2})\,x_2x_1x_0^2x_{-2} + \dots\dots \\
& + (21\bar{1}^2)\,x_2x_1x_{-1}^2 + (210\bar{1}^2)\,x_2x_1x_0x_{-1}^2 + (210^2\bar{1}^2)\,x_2x_1x_0^2x_{-1}^2 + \dots\dots \\
& + (1^3\bar{2})\,x_1^3x_{-2} + (1^30\bar{2})\,x_1^3x_0x_{-2} + (1^30^2\bar{2})\,x_1^3x_0^2x_{-2} + \dots\dots \\
& + (1^3\bar{1}^2)\,x_1^3x_{-1}^2 + (1^30\bar{1}^2)\,x_1^3x_0x_{-1}^2 + (1^30^2\bar{1}^2)\,x_1^3x_0^2x_{-1}^2 + \dots\dots \\
& + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

$$
\begin{aligned}
X_{-2} = {} & (\bar{2})\,x_{-2} + (0\bar{2})\,x_0x_{-2} + (0^2\bar{2})\,x_0^2x_{-2} + \dots\dots \\
& + (1\bar{3})\,x_1x_{-3} + (10\bar{3})\,x_1x_0x_{-3} + (10^2\bar{3})\,x_1x_0^2x_{-3} + \dots\dots \\
& + (\bar{1}^2)\,x_{-1}^2 + (0\bar{1}^2)\,x_0x_{-1}^2 + (0^2\bar{1}^2)\,x_0^2x_{-1}^2 + \dots\dots \\
& + (1\overline{12})\,x_1x_{-1}x_{-2} + (10\overline{12})\,x_1x_0x_{-1}x_{-2} + (10^2\overline{12})\,x_1x_0^2x_{-1}x_{-2} + \dots\dots \\
& + (1\bar{1}^3)\,x_1x_{-1}^3 + (10\bar{1}^3)\,x_1x_0x_{-1}^3 + (10^2\bar{1}^3)\,x_1x_0^2x_{-1}^3 + \dots\dots \\
& + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

we have

$$X_1X_{-2} = \dots\dots + \{(1^2\bar{1})(0\bar{2}) + (1^20\bar{1})(\bar{2}) + (1)(10\overline{12}) + (10)(1\overline{12})\}\,x_1^2x_0x_{-1}x_{-2} + \dots\dots$$

79. The partition of the term $x_1^2x_0x_{-1}x_{-2}$ is $(1^20\overline{12})$; each of the products

$$(1^2\bar{1})(0\bar{2}), \quad (1^20\bar{1})(\bar{2}), \quad (1)(10\overline{12}), \quad (10)(1\overline{12}),$$

is a separation of the partition $(1^20\overline{12})$ of specification $(1\bar{2})$; this follows of necessity because $(1\bar{2})$ is the partition of the term X_1X_{-2}.

80. When the products which occur in the coefficient of the term $x_1^2x_0x_{-1}x_{-2}$ are multiplied, a monomial symmetric function $(1^2\overline{12})$ will be presented attached to a certain numerical coefficient; supposing the symmetric functions refer to quantities $a, b, c, \dots\dots$, we have

$$(1^2\bar{1})(0\bar{2}) = \sum \frac{a}{1} \cdot \frac{b}{1} \cdot \frac{1}{c} \sum \frac{a}{a} \cdot \frac{1}{b^2},$$

and also

$$(1^2\overline{12}) = \sum \frac{a}{1} \cdot \frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}.$$

81. A term $\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ of the symmetric function $(1^2\bar{1}\bar{2})$ arises from the multiplication $(1^2\bar{1})(0\bar{2})$ in the three ways:

$$\left(\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\right)\left(\frac{a}{a}\cdot\frac{1}{d^2}\right),$$

$$\left(\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\right)\left(\frac{b}{b}\cdot\frac{1}{d^2}\right),$$

$$\left(\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\right)\left(\frac{c}{c}\cdot\frac{1}{d^2}\right),$$

for each of the terms $\frac{a^2}{a}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$, $\frac{a}{1}\cdot\frac{b^2}{b}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$, $\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{c}{c^2}\cdot\frac{1}{d^2}$, is the same term $\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ of the function $(1^2\bar{1}\bar{2})$.

82. Observe that such a product as

$$\left(\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\right)\left(\frac{e}{e}\cdot\frac{1}{d^2}\right)$$

gives rise to a term $\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{e}{e}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ which belongs to the function $(1^2 0\bar{1}\bar{2})$ and not to $(1^2\bar{1}\bar{2})$; the coefficient of $(1^2\bar{1}\bar{2})$ in the product $(1^2\bar{1})(0\bar{2})$ is thus 3.

83. To connect this result with the preceding theory of distribution, observe that the terms

$$\frac{a^2}{a}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2},\quad \frac{a}{1}\cdot\frac{b^2}{b}\cdot\frac{1}{c}\cdot\frac{1}{d^2},\quad \frac{a}{1}\cdot\frac{b}{1}\cdot\frac{c}{c^2}\cdot\frac{1}{d^2},$$

may each be considered as representing an assemblage of 7 objects $(1^2\bar{1}\bar{2})$, the numerator and denominator letters denoting objects in the upper and lower groups respectively; the term $\frac{a^2}{a}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ arose from the multiplication

$$\left(\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\right)\left(\frac{a}{a}\cdot\frac{1}{d^2}\right),$$

and, conversely, we may regard it as decomposed in this manner; we may further consider this decomposition of the term $\frac{a^2}{a}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ to denote a distribution of the assemblage of 7 objects represented by the term; this distribu-

tion will be into an assemblage of 7 parcels $(1\bar{2})$ and will be indicated by the scheme

$$\begin{array}{ll} AA & B \\ ab & a \\ A & BBB \\ c & add \end{array}$$

84. Drawing a vertical line between the A and the B parcels, the scheme breaks up into two portions; the left-hand portion denotes a distribution of objects $(1^2\bar{1})$ into parcels (1), whilst the right-hand shows a distribution of objects $(0\bar{2})$ into parcels $(\bar{2})$; the distribution is of objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$, and it is necessarily subject to a restriction whose partition is $(1^20\bar{1}\bar{2})$ because the term $x_1^2x_0x_{-1}x_{-2}$ has this partition; the two portions into which the distribution may be divided are respectively restricted by partitions $(1^2\bar{1})$ and $(0\bar{2})$ because these partitions are factors of the separation

$$(1^2\bar{1})(0\bar{2})$$

which is being discussed.

85. Two more distributions of precisely the same nature correspond to the two terms

$$\frac{a}{1}\cdot\frac{b^3}{b}\cdot\frac{1}{c}\cdot\frac{1}{d^2},\quad \frac{a}{1}\cdot\frac{b}{1}\cdot\frac{c}{c^3}\cdot\frac{1}{d^2};$$

these are

$$\begin{array}{ll} AA & B \\ ab & b \\ A & BBB \\ c & bdd \end{array},\qquad \begin{array}{ll} AA & B \\ ab & c \\ A & BBB \\ c & cdd \end{array};$$

each of the three distributions is of objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$, and is not only subject to the restriction whose partition is $(1^20\bar{1}\bar{2})$, but also more minutely to the compound restriction indicated by the separation $(1^2\bar{1})(0\bar{2})$.

86. It is thus clear that, corresponding to the algebraical result

$$(1^2\bar{1})(0\bar{2}) = \ldots. + 3\,(1^2\bar{1}\bar{2}) + \ldots.,$$

we have a distribution theorem, viz.

"There are 3 ways of distributing objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$, $(1\bar{2})$ being the specification of the separation $(1^2\bar{1})(0\bar{2})$, subject to the compound restriction, of separation $(1^2\bar{1})(0\bar{2})$."

87. Consider next the product

$$(1^2 0\bar{1})(\bar{2}) = \sum \frac{a}{1} \cdot \frac{b}{1} \cdot \frac{c}{c} \cdot \frac{1}{d} \sum \frac{1}{a^2};$$

the term $\frac{a}{1} \cdot \frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}$ can only arise from the product

$$\left(\frac{a}{1} \cdot \frac{b}{1} \cdot \frac{1}{c} \cdot \frac{d}{d}\right)\left(\frac{1}{d^2}\right);$$

the coefficient of $(1^2\bar{1}\bar{2})$ in the product $(1^2 0\bar{1})(\bar{2})$ is therefore unity; the corresponding distribution is seen to be

$$\begin{array}{ll} AAA & \\ abd & \\ AA & BB \\ cd & dd \end{array};$$

the restrictions in the A and B parcels are respectively $(1^2 0\bar{1})$ and $(\bar{2})$; hence we have a distribution of objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$ subject to the composite restriction $(1^2 0\bar{1})(\bar{2})$.

88. Again the product

$$(1)(10\bar{1}\bar{2}) = \sum \frac{a}{1} \sum \frac{a}{1} \cdot \frac{b}{b} \cdot \frac{1}{c} \cdot \frac{1}{d^2};$$

the term $\frac{a}{1} \cdot \frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}$ is obtained from 2 products

$$\left(\frac{b}{1}\right)\left(\frac{a}{1} \cdot \frac{b}{b} \cdot \frac{1}{c} \cdot \frac{1}{d^2}\right),$$
$$\left(\frac{a}{1}\right)\left(\frac{a}{a} \cdot \frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}\right);$$

thus the coefficient of $(1^2\bar{1}\bar{2})$ in the product $(1)(10\bar{1}\bar{2})$ is 2, and the corresponding distributions are

$$\begin{array}{ll} A & BB \\ b & ab \\ & BBBB \\ & bcdd \end{array}, \qquad \begin{array}{ll} A & BB \\ a & ab \\ & BBBB \\ & acdd \end{array};$$

which are distributions of objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$, subject to the composite restriction $(1)(10\bar{1}\bar{2})$.

89. Finally we have the product

$$(10)(1\bar{1}\bar{2}) = \sum \frac{a}{1}\cdot\frac{b}{b}\sum\frac{a}{1}\cdot\frac{1}{b}\cdot\frac{1}{c^2};$$

the term $\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ is obtained from 6 products

$$\left(\frac{a}{1}\cdot\frac{b}{b}\right)\left(\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}\right),\quad \left(\frac{a}{a}\cdot\frac{b}{1}\right)\left(\frac{a}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}\right),$$
$$\left(\frac{a}{1}\cdot\frac{c}{c}\right)\left(\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}\right),\quad \left(\frac{b}{1}\cdot\frac{c}{c}\right)\left(\frac{a}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}\right),$$
$$\left(\frac{a}{1}\cdot\frac{d}{d}\right)\left(\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}\right),\quad \left(\frac{b}{1}\cdot\frac{d}{d}\right)\left(\frac{a}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}\right);$$

the coefficient of $(1^2\bar{1}\bar{2})$ in the product $(10)(1\bar{1}\bar{2})$ is thus 6, and the corresponding distributions are

$$\begin{array}{ll} AA & B \\ ab & b \\ A & BBB \\ b & cdd \end{array},\quad \begin{array}{ll} AA & B \\ ab & a \\ A & BBB \\ a & cdd \end{array},$$

$$\begin{array}{ll} AA & B \\ ac & b \\ A & BBB \\ c & cdd \end{array},\quad \begin{array}{ll} AA & B \\ bc & a \\ A & BBB \\ c & cdd \end{array},$$

$$\begin{array}{ll} AA & B \\ ad & b \\ A & BBB \\ d & cdd \end{array},\quad \begin{array}{ll} AA & B \\ bd & a \\ A & BBB \\ d & cdd \end{array};$$

these are distributions of objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$, subject to the composite restriction $(10)(1\bar{1}\bar{2})$.

90. Altogether, in the product X_1X_{-2}, the coefficient of $(1^2\bar{1}\bar{2})\,x_1^2x_0x_{-1}x_{-2}$ is 12 $(= 3 + 1 + 2 + 6)$; the 12 corresponding distributions have been exhibited; each of these had reference to objects $(1^2\bar{1}\bar{2})$ and parcels $(1\bar{2})$; each, further, was associated with a composite restriction which was denoted by a separation of the partition $(1^20\bar{1}\bar{2})$ because the term $x_1^20x_{-1}x_{-2}$ has this partition; each of these separations had the specification $(1\bar{2})$ because $(1\bar{2})$ is the partition of the term X_1X_{-2}; the 12 distributions were complete, that is, they included all those that

were possible under the given conditions; this must be so because there is a one-to-one correspondence between the distributions and term products, and care was taken to consider the whole of the latter. Amongst the separations which denoted composite restrictions were included all separations of $(1^2 0\bar{1}\bar{2})$ which had the specification $(1\bar{2})$; this is a consequence of the forms of the functions X and X_{-2}. Hence if we consider the whole cofactor of $x_1^2 x_0 x_{-1} x_{-2}$, which arises from the product $X_1 X_{-2}$ and therein the coefficient of $(1^2\bar{1}\bar{2})$, we find that this coefficient denotes the number of ways of distributing objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$ subject to the restriction whose partition is $(1^2 0\bar{1}\bar{2})$; this restriction does and must involve all the composite restrictions whose separations have a specification $(1\bar{2})$, and there is no need to specifically mention the circumstance in describing the distribution; we may simply state that the analytical result

$$X_1 X_{-2} = \ldots . + 12\,(1^2\bar{1}\bar{2})\,x_1^2 x_0 x_{-1} x_{-2} + \ldots .$$

is the analytical statement of the arithmetical theorem: "There are 12 ways of distributing objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$, subject to the restriction whose partition is $(1^2 0\bar{1}\bar{2})$."

91. In the case just considered there is a one-to-one correspondence between the literal products and the distributions; this, however, does not always obtain. Suppose that we take the product of symmetric functions $(1^2 0)(2)$, in which each factor is of the same weight 2, and seek the coefficient of (21^2) in its development; proceeding in the usual manner, we have

$$(1^2 0)(2) = \sum \frac{a}{1} \cdot \frac{b}{1} \cdot \frac{c}{c} \sum \frac{a^2}{1}$$

and

$$(21^2) = \sum \frac{a^2}{1} \cdot \frac{b}{1} \cdot \frac{c}{1};$$

the term $\frac{a^2}{1} \cdot \frac{b}{1} \cdot \frac{c}{1}$ arises only from the product

$$\left(\frac{b}{1} \cdot \frac{c}{1} \cdot \frac{a}{a}\right)\left(\frac{a^2}{1}\right),$$

but corresponding to this decomposition, there are two distributions of 6 objects (21^2) into 6 parcels (2^3), viz.

$$\begin{matrix} AAA & BB \\ abc & aa \\ A & \\ a & \end{matrix}\;,\qquad \begin{matrix} AA & BBB \\ aa & abc \\ & B \\ & a \end{matrix}\;;$$

the fact is that the component partitions (1^20) and (2) being of the *same* weight but *different*, we obtain an additional distribution by the interchange of A and B; but if we form the product X_2^2 we obtain a term $2(1^20)(2)$, the 2 appearing for the very reason that (1^20) and (2) are of the same weight but different; we may therefore effect a one-to-one correspondence between the literal products in $2(1^20)(2)$ and the distributions thence arising. Similarly, if we form the product X_λ^l, and $(\daleth_1)$, $\daleth_2)$, ... denote different partitions of weight λ, we will on development obtain a term which involves

$$\frac{(l_1+l_2+\ldots.)!}{l_1!\,l_2!\ldots.}(\daleth_1)^{l_1}(\daleth_2)^{l_2}\ldots.;$$

and, moreover, corresponding to a literal product in $(\daleth_1)^{l_1}(\daleth_2)^{l_2}\ldots.$ there will be precisely

$$\frac{(l_1+l_2+\ldots.)!}{l_1!\,l_2!\ldots.}$$

distributions, since we may permute the capital letters in any one distribution in all possible ways; thus we may consider that there exists a one-to-one correspondence between the literal products in

$$\frac{(l_1+l_2+\ldots.)!}{l_1!\,l_2!\ldots.}(\daleth_1)^{l_1}(\daleth_2)^{l_2}\ldots.$$

and the distributions which arise from them.

92. In general, if partitions of the same weight p_s (where p_s is positive, zero, or negative) be denoted by (P_s'), (P_s''), (P_s'''), ..., the development of the product

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots.$$

will produce a term involving

$$\frac{(\pi_1'+\pi_1''+\ldots.)!}{\pi_1'!\,\pi_1''!\ldots.}\cdot\frac{(\pi_2'+\pi_2''+\ldots.)!}{\pi_2'!\,\pi_2''!\ldots.}\ldots.(P_1)^{\pi_1'}(P_1')^{\pi_1''}\ldots.(P_2)^{\pi_2'}(P_2')^{\pi_2''}\ldots.,$$

and there will be a one-to-one correspondence between the literal terms occurring therein and the distributions arising therefrom.

93. Hence, from what has gone before, the result

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots.=\ldots.+\theta(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots.)x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots.+\ldots.$$

is the analytical statement of the arithmetical theorem: "There are θ ways of distributing objects $(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots.)$ into parcels $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots.)$, subject to the restriction whose partition is $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots.)$."

94. Recalling our former result

$$X_1X_{-2} = \ldots . + 12\,(1^2\bar{1}\bar{2})\,x_1^2x_0x_{-1}x_{-2} + \ldots . ,$$

we can now establish, in an instantaneous manner, the reciprocal result

$$X_1^2X_{-1}X_{-2} = \ldots . + 12\,(1\bar{2})\,x_1^2x_0x_{-1}x_{-2} + \ldots . ;$$

for, take any one of the foregoing 12 distributions, viz.

$$\begin{array}{ll} AA & B \\ ab & a \\ A & BBB \\ a & cdd \end{array},$$

and change the small letters into capitals and vice versa, we get thus a distribution

$$\begin{array}{ll} aa & b \\ AB & A \\ a & bbb \\ A & CDD \end{array},$$

which may be put into the form

$$\begin{array}{llll} AA & B & & \\ ab & a & & \\ A & & C & DD \\ a & & b & bb \end{array},$$

and this denotes a distribution of objects $(1\bar{2})$ into parcels $(1^2\bar{1}\bar{2})$, subject to a restriction whose partition is $(1^20\bar{1}\bar{2})$.

95. We have thus passed from a distribution of objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$ to a distribution of objects $(1\bar{2})$ into parcels $(1^2\bar{1}\bar{2})$ without altering the restriction which still possesses the partition $(1^20\bar{1}\bar{2})$.

96. This interchange of small and capital letters (in reality an interchange of objects and parcels) cannot possibly alter the partition of restriction; this is manifest from the definition of the latter.

97. Further, the process is reversible; from every distribution of the second kind we are able to pass to a distribution of the first kind and vice versa.

98. There is thus a one-to-one correspondence between the two natures of distribution, and the numbers of the distributions of the two kinds must be identical. Hence

$$X_1^2X_{-1}X_{-2} = \ldots . + 12\,(1\bar{2})\,x_1^2x_0x_{-1}x_{-2} + \ldots . ,$$

for this is merely the analytical statement of the arithmetical fact that there are 12 distributions of objects $(1\bar{2})$ into parcels $(1^2\bar{1}\bar{2})$ subject to a restriction whose partition is $(1^2 0 \bar{1}\bar{2})$.

99. The general theorem is now practically established, for if

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots . = \ldots . + \theta\,(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots .)\, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots . + \ldots . ,$$

there are θ ways of distributing objects $(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots .)$ into parcels $(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots .)$ subject to a restriction whose partition is $(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots .)$, and the above reversible process proves that there must be also exactly θ ways of distributing objects $(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots .)$ into parcels $(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots .)$, subject to the same restriction; hence

$$X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2} X_{\lambda_3}^{l_3} \ldots . = \ldots . + \theta\,(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots .)\, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots . + \ldots . ,$$

the theorem to be demonstrated.

100. This proposition is cardinal in symmetrical algebra and of great importance; I hope, in a subsequent memoir in this Journal, to give another proof of it by means of differential operators.

Section 6.

The Formation of Symmetrical Tables.

101. One of the consequences of the theorem of reciprocity is the possibility of forming a pair of tables of symmetric functions, of a symmetrical character, in association with every partition, in positive, zero, and negative integers, of every number, positive, zero, or negative.

102. For, let the separations of the partition $(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots .)$ possess in all r specifications which may be

$$\varkappa_1,\ \varkappa_2,\ \varkappa_3,\ \ldots .\ \varkappa_r,$$

and let, moreover,

$$[X_{\varkappa_1}],\ [X_{\varkappa_2}],\ [X_{\varkappa_3}],\ \ldots .\ [X_{\varkappa_r}]$$

denote the corresponding X-products, so that if

$$\varkappa_m = (\mu_1 \mu_2 \mu_3 \ldots .),$$
$$[X_{\varkappa_m}] = X_{\mu_1} X_{\mu_2} X_{\mu_3} \ldots .$$

103. The law of reciprocity shows that if

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots . = \ldots . + P x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots . + \ldots . ,$$

so that P consists of an assemblage of separations of the partition $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots)$, each of which has the specification $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)$ which is one of the series of specifications

$$\varkappa_1, \varkappa_2, \varkappa_3, \ldots \varkappa_r,$$

P on development will only give rise to symmetric functions which are symbolized by partitions included in the specification set $\varkappa_1, \varkappa_2, \varkappa_3, \ldots \varkappa_r$; for otherwise the law of reciprocity could not be true.

104. Now form X-products corresponding to all the specifications; let $P_{\kappa_1}, P_{\kappa_2}, P_{\kappa_3}, \ldots P_{\kappa_r}$ be the corresponding values of P, and further let $\theta_{t,m}$ be the numerical coefficient of $\varkappa_m x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots$ in the development of $[X_{\kappa_t}]$, or what is the same thing, the coefficient of $\varkappa_m$ in the development of P_{κ_t}; thus,

$$\begin{aligned}
X_{\kappa_1} &= \ldots + P_{\kappa_1}x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots \\
&= \ldots + (\theta_{1,1}\varkappa_1 + \theta_{1,2}\varkappa_2 + \theta_{1,3}\varkappa_3 + \ldots + \theta_{1,r}\varkappa_r)\, x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots, \\
X_{\kappa_2} &= \ldots + P_{\kappa_2}x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots \\
&= \ldots + (\theta_{21}\varkappa_1 + \theta_{22}\varkappa_2 + \theta_{23}\varkappa_3 + \ldots + \theta_{2r}\varkappa_r)\, x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots, \\
X_{\kappa_3} &= \ldots + P_{\kappa_3}x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots \\
&= \ldots + (\theta_{31}\varkappa_1 + \theta_{32}\varkappa_2 + \theta_{33}\varkappa_3 + \ldots + \theta_{3r}\varkappa_r)\, x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
X_{\kappa_r} &= \ldots + P_{\kappa_r}x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots \\
&= \ldots + (\theta_{r1}\varkappa_1 + \theta_{r2}\varkappa_2 + \theta_{r3}\varkappa_3 + \ldots + \theta_{rr}\varkappa_r)\, x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots
\end{aligned}$$

105. This result shows that the assemblages of separations

$$P_{\kappa_1}, P_{\kappa_2}, P_{\kappa_3}, \ldots P_{\kappa_r}$$

are linearly connected with the specifications

$$\varkappa_1, \varkappa_2, \varkappa_3, \ldots \varkappa_r,$$

and that we may form a table, viz.

	$\varkappa_1$	$\varkappa_\kappa$	$\varkappa_3$	$\ldots$	$\varkappa_r$
P_{κ_1}	θ_{11}	θ_{12}	θ_{13}	$\ldots$	θ_{1r}
P_{κ_2}	θ_{21}	θ_{22}	θ_{23}	$\ldots$	θ_{2r}
P_{κ_3}	θ_{31}	θ_{32}	θ_{33}	$\ldots$	θ_{3r}
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
P_{κ_r}	θ_{r1}	θ_{r2}	θ_{r3}	$\ldots$	θ_{rr}

and then by the law of reciprocity

$$\theta_{pq} = \theta_{qp},$$

and the table enjoys row and column (i. e. diagonal) symmetry.

106. We may similarly invert the table and express $x_1, x_2, x_3, \ldots x_r$ as linear functions of $P_{\kappa_1}, P_{\kappa_2}, P_{\kappa_3}, \ldots P_{\kappa_r}$ in a table enjoying the same symmetry.

107. To make the meaning clear, omit in the first instance all partitions which contain zero or negative parts, and write down a complete system of X-products for any given weight, as follows, e. g. weight $= 5$:

	5 X_5	41 X_4X_1	32 X_3X_2	31^2 $X_3X_1^2$	2^21 $X_2^2X_1$	21^3 $X_2X_1^3$	1^5 X_1^5
x_5	(5)						
x_4x_1	(41)	$(4)(1)$					
x_3x_2	(32)		$(3)(2)$				
$x_3x_1^2$	(31^2)	$(31)(1)$	$(3)(1^2)$	$(3)(1)^2$			
$x_2^2x_1$	(2^21)	$(2^2)(1)$	$(2)(21)$		$(2)^2(1)$		
$x_2x_1^3$	(21^3)	$(21^2)(1)$	$(2)(1^3)+(21)(1^2)$	$(21)(1)^2$	$2\,(2)(1^2)(1)$	$(2)(1)^3$	
x_1^5	(1^5)	$(1^4)(1)$	$(1^3)(1^2)$	$(1^3)(1)^2$	$(1^2)^2(1)$	$(1^2)(1)^3$	$(1)^5$

here each line is a set of "assemblages of separations," each assemblage having its own specification, as appearing by the top line. The assemblages and specifications represent symmetric functions, and the theorem is that these symmetric functions are linearly connected, the coefficients being symmetrical in regard to a diagonal. Thus, from the last line but one we have the assemblages (separations of (21^3))

$$(21^3),\ (21^2)(1),\ (2)(1^3)+(21)(1^2),\ (21)(1)^2,\ 2\,(2)(1^2)(1),\ (2)(1)^3$$

linearly connected with the specifications

$$(5), \quad (41), \quad (32), \quad (31^2), \quad (2^21), \quad (21^3).$$

108. Again, let us take the weight -2 and the separable partition $(0^2\bar{1}^2)$; the corresponding portion of the table of X-products is

	$(\bar{2})$ X_{-2}	$(0\bar{2})$ X_0X_{-2}	$(0^2\bar{2})$ $X_0^2X_{-2}$	$(\bar{1}^2)$ X_{-1}^2	$(0\bar{1}^2)$ $X_0X_{-1}^2$	$(0^2\bar{1}^2)$ $X_0^2X_{-1}^2$
$x_0^2x_{-1}^2$	$(0^2\bar{1}^2)$	$(0)(0\bar{1}^2)+(0^2)(\bar{1}^2)$	$(0)^2(\bar{1}^2)$	$2(0^2\bar{1})(\bar{1})+(0\bar{1})^2$	$(0^2)(\bar{1})^2+2(0)(0\bar{1})(\bar{1})$	$(0)^2(\bar{1})^2$

showing that we have the assemblages (separations of $(0^2\bar{1}^2)$) indicated in the bottom line, linearly connected with the specifications shown in the top line.

109. Writing down the assemblages in a vertical column and the specifications in a horizontal row, we may then form a table which calculation shows to be

	$(\bar{2})$	$(0\bar{2})$	$(0^2\bar{2})$	$(\bar{1}^2)$	$(0\bar{1}^2)$	$(0^2\bar{1}^2)$
$(0^2\bar{1}^2)$						1
$(0)(0\bar{1}^2)+(0^2)(\bar{1}^2)$				1	5	3
$(0)^2(\bar{1}^2)$				4	5	2
$2(0^2\bar{1})(\bar{1})+(0\bar{1})^2$		1	4	2	10	8
$(0^2)(\bar{1})^2+2(0)(0\bar{1})(\bar{1})$		5	5	10	20	10
$(0)^2(\bar{1})^2$	1	3	2	8	10	4

where the third line is to be read

$$(0)^2(\bar{1}^2)=4(\bar{1}^2)+5(0\bar{1}^2)+2(0^2\bar{1}^2),$$

or in an algebraic form

$$\left(\sum\alpha^0\right)^2\sum\alpha^{-1}\beta^{-1}=4\sum\alpha^{-1}\beta^{-1}+5\sum\alpha^0\beta^{-1}\gamma^{-1}+2\sum\alpha^0\beta^0\gamma^{-1}\delta^{-1},$$

verified through the medium of the identity

$$n^2 \qquad =4 \qquad +5(n-2) \qquad +2.\tfrac{1}{2}(n-2)(n-3).$$

110. The table already given possesses diagonal symmetry as a direct consequence of the law of reciprocity; the inverse table, which expresses the specifi-

cations as linear functions of the assemblages of separations, necessarily enjoys the same symmetry. Its form is

	$(0^2\bar{1}^2)$	$(0)(0\bar{1}^2) + (0^2)(\bar{1}^2)$	$(0)^2(\bar{1}^2)$	$2(0^2\bar{1})(\bar{1}) + (0\bar{1})^2$	$(0^2)(\bar{1})^2 + 2(0)(0\bar{1})(\bar{1})$	$(0)^2(\bar{1})^2$
$(\bar{2})$	$-\frac{2}{3}$	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	1
$(0\bar{2})$	$\frac{2}{3}$	$\frac{2}{15}$	$-\frac{8}{15}$	$-\frac{1}{3}$	$\frac{4}{15}$	
$(0^2\bar{2})$	$-\frac{2}{3}$	$-\frac{8}{15}$	$\frac{8}{15}$	$\frac{1}{3}$	$-\frac{1}{15}$	
$(\bar{1}^2)$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$			
$(0\bar{1}^2)$	$-\frac{2}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$			
$(0^2\bar{1}^2)$	1					

111. It has thus been demonstrated that a pair of symmetrical tables exist in the case of every partition into positive, zero, and negative integers of every number positive, zero, or negative.

112. The theorem in regard to the coefficients in a group, given on page 35 of the former memoir, is extended easily to this enlarged theory, and we may enunciate as follows:

113. *Theorem.* "In the expression of symmetric function

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots)$$

by means of separations of

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \dots),$$

where the parts of the partitions are positive, zero, or negative, the algebraic sum of the coefficients in each group will be zero if the partition

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots)$$

possesses no separation of specification

$$(\sigma_1 s_1,\ s_2\sigma_2,\ s_3\sigma_3,\ \ldots\ldots).\text{"}$$

114. This theorem may be verified in the second of the tables above given in the cases of $(\bar{2})$ and $(\bar{1}^2)$ only, as all the other symmetric functions in the left-hand vertical column possess separations of specification $(0\bar{2})$. Ex. gr.

$$\begin{aligned}(\bar{2}) = (0)^2(\bar{1})^2 \quad &- \tfrac{2}{3}(0^2)(\bar{1})^2 - \tfrac{2}{3}(0)^2(\bar{1}^2) + \tfrac{2}{3}(0^2)(\bar{1}^2) \\ &- \tfrac{4}{3}(0)(0\bar{1})(\bar{1}) + \tfrac{2}{3}(0^2\bar{1})(\bar{1}) + \tfrac{2}{3}(0)(0\bar{1}^2) - \tfrac{2}{3}(0^2\bar{1}^2) \\ &+ \tfrac{1}{3}(0\bar{1})^2 \\ \hline &\pm\tfrac{4}{3} \qquad \pm\tfrac{2}{3} \qquad \pm\tfrac{2}{3} \qquad \pm\tfrac{2}{3}. \\ \hline\end{aligned}$$

$$\begin{aligned}(\bar{1}^2) = \tfrac{1}{3}(0)^2(\bar{1}^2) &- \tfrac{1}{3}(0^2)(\bar{1}^2) \\ - \tfrac{1}{3}(0)(0\bar{1}^2) &+ \tfrac{1}{3}(0^2\bar{1}^2) \\ \hline \pm\tfrac{1}{3} \qquad &\pm\tfrac{1}{3}. \\ \hline\end{aligned}$$

SECTION 7.

The Law of Expressibility.

115. The law given on page 6 of former memoir may now be extended as follows:

116. *Theorem.* "If a symmetric function be symbolized by

$$(\lambda\mu\nu\ \ldots\ldots)$$

the parts λ, μ, ν, ... being positive, zero, or negative, and

$$\begin{array}{llll} (\lambda_1\lambda_2\lambda_3\ \ldots\ldots) & \text{be any partition of } \lambda, \\ (\mu_1\mu_2\mu_3\ \ldots\ldots) & \text{"} \quad \text{"} \quad \text{"}\ \mu, \\ (\nu_1\nu_2\nu_3\ \ldots\ldots) & \text{"} \quad \text{"} \quad \text{"}\ \nu, \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{array}$$

the symmetric function $\qquad (\lambda\mu\nu\ \ldots\ldots)$

is expressible as a linear function of separations of

$$\lambda_1\lambda_2\lambda_3\ \ldots\ldots\ \mu_1\mu_2\mu_3\ \ldots\ \nu_1\nu_2\nu_3\ \ldots\ldots).\text{"}$$

117. As an example of this, we may express the function (0^3) as a linear

function of separations of (0^4); it will be interesting to give, as well, the complete tables of separations of (0^4) which includes this result.

	(0)	(0^2)	(0^3)	(0^4)
$(0)^4$	1	14	36	24
$3(0^2)(0)^2$		12	45	36
$(0^2)^2 + 2(0^3)(0)$		1	12	14
(0^4)				1

	$(0)^4$	$3(0^2)(0)^2$	$(0^2)^2 + 2(0^3)(0)$	(0^4)
(0)	1	$-\frac{4}{3}$	2	-4
(0^2)		$\frac{4}{33}$	$-\frac{5}{11}$	2
(0^3)		$-\frac{1}{99}$	$\frac{4}{33}$	$-\frac{4}{3}$
(0^4)				1

from which

$$(0^2) = \tfrac{4}{33} \cdot 3(0^2)(0)^2 - \tfrac{5}{11}\{(0^2)^2 + 2(0^3)(0)\} + 2(0^4);$$

and this merely exhibits a relation connecting the second, third, fourth and fifth binomial coefficients in the expansion of $(1+x)^n$; for

$$(1+x)^n = 1 + (0)x + (0^2)x^2 + (0^3)x^3 + (0^4)x^4 + \ldots + (0^n)x^n.$$

118. The subject of "Expansion by factorials," which is usually discussed in works on Finite Differences, is thus clearly within the domain of this theory, and the two tables last given might have been expressed by the notation and symbols of the calculus of Finite Differences.

On this subject I hope to say more upon a future occasion.

119. It is, in conclusion, to be particularly observed that all algebra is expressible by means of factorials, and thus any algebraical expression whatever, of a finite nature, may be exhibited as a symmetric function of one or more sets of quantities.

ROYAL ARSENAL, WOOLWICH, ENGLAND, December 1st, 1888.

De l'homographie en mécanique.

Par P. Appell.

"La découverte des *principes de projection centrale* marque incontestablement une époque importante dans l'histoire de la géomètrie moderne. Les méthodes fondées sur ces principes possèdent un caractère à la fois intuitif et systématique, qui les rend également propres à découvrir de nouvelles propriétés des figures et à rattacher tout un ensemble de propositions à une même vérité générale." *

Il nous a paru intéressant de montrer que ces mêmes principes peuvent être appliqués, en mécanique, au mouvement d'un ou de plusieurs points libres sollicités par des forces qui ne dépendent que des positions des points. On peut, par exemple, à l'aide de la transformation homographique, rattacher les unes aux autres des questions de mécanique en apparence différentes, comme le mouvement d'un point attiré par un centre fixe proportionnellement à la distance et le mouvement d'un point attiré par un plan fixe en raison inverse du cube de la distance.

Un cas particulier de la transformation qui fait l'objet de cette étude a été indiqué par M. Halphen (Bulletin de la Société Philomathique, 7[ième] série, t. I, page 89). Nous avons résumé les points principaux de ce travail dans une Note présentée à l'Académie des Sciences de Paris (Comptes Rendus, Séance du 4 février 1889).

1). Prenons d'abord le cas le plus simple et considérons, dans un plan fixe xOy, un point matériel de masse m sollicité par une force F dont les projections X et Y sur les axes Ox et Oy sont des fonctions des seules coordonnées x et y du point; les équations du mouvement seront

$$m\frac{d^2x}{dt^2} = X, \quad m\frac{d^2y}{dt^2} = Y. \tag{1}$$

---

* Voyez une Note de M. Moutard, *Applications d'Analyse et de Géométrie* de Poncelet. Tome I, page 509.

Supposons que l'on ait trouvé des expressions de x et y en fonction de t vérifiant ces équations, et faisons la transformation homographique définie par les formules

$$x_1 = \frac{ax + by + c}{a''x + b''y + c''}, \quad y_1 = \frac{a'x + b'y + c'}{a''x + b''y + c''} \tag{2}$$

en remplaçant en même temps la variable indépendante t par une autre variable t_1 liée à t par la relation

$$kdt_1 = \frac{dt}{(a''x + b''y + c'')^2}, \tag{3}$$

où k désigne une constante différente de zéro. Nous supposons que le déterminant

$$\Delta = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$$

n'est pas nul, et nous désignons par A, B, C, A', B', C', A'', B'', C'' les coefficients respectifs de a, b, c, a', b', c', a'', b'', c'' dans le développement de ce déterminant; ainsi

$$A = b'c'' - c'b'', \quad B = c'a'' - a'c'', \quad C = a'b'' - b'a'',$$
$$A' = cb'' - bc'', \ldots \text{ etc.}, \ldots$$

Cela posé, on a

$$\frac{dx_1}{dt_1} = k(a''x + b''y + c'')^2 \frac{dx_1}{dt}$$
$$= k\left[\left(a\frac{dx}{dt} + b\frac{dy}{dt}\right)(a''x + b''y + c'') - \left(a''\frac{dx}{dt} + b''\frac{dy}{dt}\right)(ax + by + c)\right]$$

ou, en réduisant

$$\frac{dx_1}{dt_1} = k\left[\quad C'\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) + B'\frac{dx}{dt} - A'\frac{dy}{dt}\right];$$

on trouve de même

$$\frac{dy_1}{dt_1} = k\left[-C\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) - B\frac{dx}{dt} + A\frac{dy}{dt}\right].$$

On tire de là

$$\frac{d^2x_1}{dt_1^2} = k^2(a''x + b''y + c'')^3\left[\quad C'\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) + B'\frac{d^2x}{dt^2} - A'\frac{d^2y}{dt^2}\right],$$
$$\frac{d^2y_1}{dt_1^2} = k^2(a''x + b''y + c'')^3\left[-C\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) - B\frac{d^2x}{dt^2} + A\frac{d^2y}{dt^2}\right];$$

ce que l'on peut écrire, d'après les équations (1),

$$m\frac{d^2x_1}{dt_1^2}=X_1, \quad m\frac{d^2y_1}{dt_1^2}=Y_1, \tag{4}$$

en posant

$$\left.\begin{aligned} X_1 &= k^3(a''x+b''y+c'')^3[\quad C'(xY-yX)+B'X-A'Y] \\ Y_1 &= k^3(a''x+b''y+c'')^3[-C(xY-yX)-BX\ +AY] \end{aligned}\right\}. \tag{5}$$

Par hypothèse X et Y sont des fonctions de x et y; X_1 et Y_1 sont donc des fonctions de x et y que l'on pourra toujours, à l'aide des formules de transformation (2), exprimer en fonction de x_1 et y_1. Il suffira pour cela de résoudre les équations (2) par rapport à x et y, ce qui donne

$$x=\frac{Ax_1+A'y_1+A''}{Cx_1+C'y_1+C''}, \quad y=\frac{Bx_1+B'y_1+B''}{Cx_1+C'y_1+C''}, \tag{6}$$

$$a''x+b''y+c''=\frac{\Delta}{Cx_1+C'y_1+C''},$$

et de porter ensuite ces valeurs de x et y dans les expressions (5).

D'après les équations (4) le point (x_1, y_1) se meut dans le temps t_1 comme un point matériel de masse m sollicité par une force F_1 dont les projections X_1 et Y_1 dépendent seulement des coordonnées x_1 et y_1 du point. On obtient ainsi le théorême suivant:

Toutes les fois que l'on sait trouver le mouvement d'un point (x, y) sous l'action d'une force F dépendant seulement de la position du mobile, on en déduit, par la transformation précédente, le mouvement d'un autre point (x_1, y_1) sollicité par une force F_1 dépendant seulement de la position du mobile; la trajectoire du second point est la transformée homographique de la trajectoire du premier.

On remarquera que la droite suivant laquelle est dirigée la force F_1 est la transformée homographique de la droite suivant laquelle est dirigée la force F. En effet, en désignant par ξ_1 et η_1 les coordonnées courantes, on a pour l'équation de la droite suivant laquelle est dirigée la force F_1,

$$\frac{\xi_1-x_1}{X_1}=\frac{\eta_1-y_1}{Y_1};$$

la transformée homographique de cette droite s'obtient en remplaçant ξ_1 et η_1 par

$$\frac{a\xi+b\eta+c}{a''\xi+b''\eta+c''}, \quad \frac{a'\xi+b'\eta+c'}{a''\xi+b''\eta+c''},$$

x_1 et y_1 par les expressions (2), X_1 et Y_1 par les expressions (5). On a ainsi, après quelques réductions,

$$\frac{C'[x(\eta-y)-y(\xi-x)]+B'(\xi-x)-A'(\eta-y)}{C'(xY-yX)+B'X-A'Y}$$
$$=\frac{-C[x(\eta-y)-y(\xi-x)]-B(\xi-x)+A(\eta-y)}{-C(xY-yX)-BX+AY};$$

or cette équation, où ξ, η sont les coordonnées courantes, représente une droite passant par le point $\xi=x$, $\eta=y$ et ayant pour coefficient angulaire $\frac{Y}{X}$; c'est donc la droite suivant laquelle est dirigée la force F.

2. *Applications.* Examinons le cas particulier où la force F est *centrale*, c'est-à-dire passe par un point fixe que l'on peut toujours prendre pour origine O des coordonnées x et y; alors la force F_1 passe aussi par un point fixe ayant pour coordonnées

$$x_1'=\frac{c}{c''}, \quad y_1'=\frac{c'}{c''}$$

situé à distance finie ou infinie.

Supposons d'abord c'' différent de zéro; le point (x_1', y_1') est alors à distance finie et on peut le prendre pour origine O_1 des coordonnées (x_1, y_1), ce qui revient à supposer

$$c=c'=0,$$

et par suite

$$A''=B''=0,$$
$$A=b'c'', \quad B=-a'c'', \quad A'=-bc'', \quad B'=ac''.$$

Appelons r la distance du mobile (x, y) à l'origine O et r_1 la distance du mobile (x_1, y_1) à l'origine O_1:

$$r=\sqrt{x^2+y^2}, \quad r_1=\sqrt{x_1^2+y_1^2}.$$

On aura, en appelant F la valeur algébrique de la force centrale (X, Y) prise positivement si la force est répulsive et négativement si elle est attractive,

$$X=F\frac{x}{r}, \quad Y=F\frac{y}{r};$$

et avec les mêmes conventions,

$$X_1=F_1\frac{x_1}{r_1}, \quad Y_1=F_1\frac{y_1}{r_1}.$$

Les formules (5) deviennent alors

$$F_1 \frac{x_1}{r_1} = k^2 (a''x + b''y + c'')^2 c'' (ax + by) \frac{F}{r},$$

$$F_1 \frac{y_1}{r_1} = k^2 (a''x + b''y + c'')^2 c'' (a'x + b'y) \frac{F}{r},$$

d'où, en vertu des relations

$$x_1 = \frac{ax + by}{a''x + b''y + c''}, \quad y_1 = \frac{a'x + b'y}{a''x + b''y + c''},$$

$$\frac{F_1}{r_1} = k^2 c'' (a''x + b''y + c'')^3 \frac{F}{r}, \tag{7}$$

formule qui exprime l'une des forces centrales en fonction de l'autre.

Exemples.—I. Si la force F est proportionnelle à la distance, $F = \mu r$, le point (x, y) décrit une conique ayant pour centre le point O ; on a alors

$$F_1 = \frac{\mu_1 r_1}{(Cx_1 + C'y_1 + C'')^3},$$

μ_1 désignant une constante, comme il résulte de la formule

$$a''x + b''y + c'' = \frac{\Delta}{Cx_1 + C'y_1 + C''}.$$

Donc un point (x_1, y_1) sollicité par une force centrale d'expression F_1 décrira une conique telle que la polaire de l'origine O_1 par rapport à cette conique soit une droite fixe

$$Cx_1 + C'y_1 + C'' = 0$$

homologue de la droite de l'infini dans le plan xOy.

II. Si la force F est en raison inverse du carré de la distance $F = \frac{\mu}{r^2}$, le point (x, y) décrit une conique ayant pour foyer le point O ; on a alors

$$F_1 = \frac{\mu_1 r_1}{(Cx_1 + C'y_1 + C'')^3 r^3},$$

et comme

$$r^2 = x^2 + y^2 = \frac{c''^2}{(Cx_1 + C'y_1 + C'')^2} [(bx_1 - ay_1)^2 + (b'x_1 - a'y_1)^2],$$

on a

$$F_1 = \frac{\mu' r_1}{[(bx_1 - ay_1)^2 + (b'x_1 - a'y_1)^2]^{\frac{3}{2}}},$$

μ' désignant une constante. Donc un point (x_1, y_1) sollicité par une force cen-

trale exprimée par cette dernière formule décrira une conique tangente à deux droites fixes distinctes

$$(bx_1 - ay_1)^2 + (b'x_1 - a'y_1)^2$$

homologues des droites isotropes

$$x^2 + y^2 = 0,$$

dans le plan xOy.

Les deux lois de force ainsi trouvées sont celles que MM. Darboux et Halphen ont signalées, à la suite d'un problème posé par M. Bertrand, comme étant les plus générales qui font décrire à leur point d'application une conique quelles que soient les conditions initiales (Comptes Rendus des Séances de l'Académie des Sciences de Paris, Tome 84). La transformation que nous venons de faire, pour déduire ces lois de l'attraction proportionnelle à la distance ou inversement proportionnelle au carré de la distance, a déjà été indiqué par M. Halphen dans le Bulletin de la Société Philomathique de Paris (7ième série, Tome I, page 89).

Dans ce qui précéde, nous avons supposé c'' différent de zéro; si l'on a

$$c'' = 0,$$

le point $x_1' = \frac{c}{c''}$, $y_1' = \frac{c'}{c''}$ par lequel passe la direction de la force F_1 est rejeté à l'infini, et la force F_1 est parallèle à une direction fixe que l'on peut toujours prendre pour direction de l'axe O_1y_1. L'on aura alors

$$c = c'' = 0, \quad c' \gtrless 0,$$
$$A' = B' = 0,$$
$$A = -b''c', \quad B = a''c', \quad A'' = bc', \quad B'' = -ac',$$

et les expressions (5) de X_1 et Y_1 deviendront

$$X_1 = 0, \quad Y_1 = -k^2(a''x + b''y)^3 c' \frac{F}{r}.$$

Exemples.—III. Si F est proportionnel à r, $F = \mu r$, on a

$$X_1 = 0, \quad Y_1 = \frac{\mu_1}{(Cx_1 + C'y_1 + C'')^3},$$

puisque

$$a''x + b''y + c'' = \frac{\Delta}{Cx_1 + C'y_1 + C''}.$$

Donc un point (x_1, y_1) sollicité par une force parallèle à l'axe O_1y_1 ayant pour expression la valeur ci-dessus de Y_1, décrira une conique dans laquelle le diamètre conjugué de la direction O_1y_1 est la droite fixe

$$Cx_1 + C'y_1 + C'' = 0.$$

Si l'on suppose $C = 0$, on a

$$Y_1 = \frac{\mu_1}{(C'y_1 + C'')^3};$$

il existe alors une fonction des forces pour la force F car

$$Fdr = \mu r dr$$

est une différentielle exacte, et il en existe une aussi pour la force transformée F_1 car

$$X_1 dx_1 + Y_1 dy_1 = \frac{\mu_1 dy_1}{(C'y_1 + C'')^3}$$

est aussi une différentielle exacte. Cet exemple montre qu'il peut exister une fonction des forces pour la force F et en même temps pour la transformée F_1.

IV. Si F est inversement proportionnelle à r^3

$$F = \frac{\mu}{r^3},$$

on a
$$X_1 = 0, \quad Y_1 = \frac{\mu_1}{(Cx_1 + C'y_1 + C'')^3} \frac{1}{r^3};$$

or
$$r^2 = x^2 + y^2 = \frac{c'^2 [(b''x_1 - b)^2 + (a''x_1 - a)^2]}{(Cx_1 + C'y_1 + C'')^2},$$

on a donc
$$X_1 = 0, \quad Y_1 = \frac{\mu'}{[(b''x_1 - b)^2 + (a''x_1 - a)^2]^{\frac{3}{2}}}.$$

Ainsi un point sollicité par une force parallèle à un axe $O_1 y_1$ ayant pour expression la valeur précédente de Y_1 décrit une conique tangente aux deux droites fixes parallèles ayant pour équations

$$(b''x_1 - b)^2 + (a''x_1 - a)^2 = 0.$$

V. On vérifiera de même que, si la force F est constante en grandeur et en direction, la force F_1 est centrale et a pour expression, si l'on prend pour origine O_1 le centre des forces F_1,

$$F_1 = \frac{\mu' r_1}{(Cx_1 + C'y_1)^3}.$$

Comme le point (x, y) sollicité par une force constante en grandeur et direction décrit une parabole dont l'axe est parallèle à la force, le point (x_1, y_1) sollicité par une force centrale d'expression F_1 décrira une conique passant par O_1 et tangente à la droite fixe

$$Cx_1 + C'y_1 = 0;$$

cette loi de force centrale est un cas limite de la seconde loi de force centrale signalée par MM. Darboux et Halphen, à savoir le cas où le trinôme homogène

du second degré en x_1, y_1, qui se trouve élevé à la puissance $\frac{3}{2}$ au dénominateur dans l'expression de cette loi de force, deviendrait un *carré parfait*.

3). Le succès de la transformation homographique que nous venons d'effectuer, conduit à essayer, comme en géométrie, des transformations plus générales obtenues en posant

$$x_1 = \phi(x, y), \quad y_1 = \psi(x, y)$$

et en faisant le changement de variable indépendante

$$dt_1 = \lambda(x, y)\, dt,$$

ϕ, ψ et λ désignant des fonctions des coordonnées x et y. L'on aura

$$\frac{dx_1}{dt_1} = \frac{1}{\lambda}\frac{dx_1}{dt} = \frac{1}{\lambda}\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{1}{\lambda}\frac{\partial\phi}{\partial y}\frac{dy}{dt},$$

$$\begin{aligned}\frac{d^2x_1}{dt_1^2} = \frac{1}{\lambda^2}\frac{\partial\phi}{\partial x}\frac{d^2x}{dt^2} + \frac{1}{\lambda^2}\frac{\partial\phi}{\partial y}\frac{d^2y}{dt^2} \\ + \frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial\phi}{\partial x}\right)\left(\frac{dx}{dt}\right)^2 + \frac{1}{\lambda}\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\frac{\partial\phi}{\partial y}\right)\left(\frac{dy}{dt}\right)^2 \\ + \frac{1}{\lambda}\left[\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial\phi}{\partial y}\right)\right]\frac{dx}{dt}\frac{dy}{dt},\end{aligned}$$

et pour $\frac{d^2y_1}{dt_1^2}$ une expression analogue déduite de la précédente en remplaçant ϕ par ψ. Les équations du mouvement

$$m\frac{d^2x}{dt^2} = X; \quad m\frac{d^2y}{dt^2} = Y$$

deviennent alors

$$m\frac{d^2x_1}{dt_1^2} = X_1, \quad m\frac{d^2y_1}{dt_1^2} = Y_1,$$

où

$$\begin{aligned}X_1 = \frac{1}{\lambda^2}\frac{\partial\phi}{\partial x}X + \frac{1}{\lambda^2}\frac{\partial\phi}{\partial y}Y + \frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial\phi}{\partial x}\right)\left(\frac{dx}{dt}\right)^2 \\ + \frac{1}{\lambda}\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\frac{\partial\phi}{\partial y}\right)\left(\frac{dy}{dt}\right)^2 + \frac{1}{\lambda}\left[\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial\phi}{\partial y}\right)\right]\frac{dx}{dt}\frac{dy}{dt},\end{aligned}$$

Y_1 étant donné par une expression analogue déduite de la précédente par le changement de ϕ en ψ. Le point (x_1, y_1) se meut donc, dans le temps t_1, comme un mobile sollicité par une force F_1 de projections X_1 et Y_1. Mais, la force F étant supposée dépendre uniquement *de la position* du mobile (x, y), cette force F_1 dépendra en général de *la position et de la vitesse* du mobile (x_1, y_1).

Cherchons quelle doit être la transformation pour que, quelle que soit la force F, la force F_1 ne dépende que de la *position* du mobile (x_1, y_1). Pour cela, il faut et il suffit que les expressions de X_1 et Y_1 ne contiennent pas les derivées premières $\frac{dx}{dt}$, $\frac{dy}{dt}$. On a donc les équations

$$\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial \phi}{\partial x}\right)=0, \quad \frac{\partial}{\partial y}\left(\frac{1}{\lambda}\frac{\partial \phi}{\partial y}\right)=0,$$
$$\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial \phi}{\partial y}\right)=0$$

et trois autres équations analogues pour la fonction ψ. Les deux premières équations précédentes montrent que

$$\frac{1}{\lambda}\frac{\partial \phi}{\partial x}=f(y), \quad \frac{1}{\lambda}\frac{\partial \phi}{\partial y}=g(x),$$

$f(y)$ étant une fonction de y seul et $g(x)$ une fonction de x seul. La troisième équation donne alors

$$f'(y)+g'(x)=0$$

relation qui ne peut avoir lieu que si $f'(y)$ et $g'(x)$ sont constants:

$$f'(y)=\alpha, \quad g'(x)=-\alpha$$

d'où
$$f(y)=\alpha y+\beta, \quad g(x)=-\alpha x-\gamma.$$

On a donc

$$\frac{\partial \phi}{\partial x}=\lambda(\alpha y+\beta), \quad \frac{\partial \phi}{\partial y}=-\lambda(\alpha x+\gamma)$$

d'où, en éliminant ϕ à l'aide de l'identité

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right),$$
$$2\lambda\alpha+\frac{\partial \lambda}{\partial y}(\alpha y+\beta)+\frac{\partial \lambda}{\partial x}(\alpha x+\gamma)=0. \tag{8}$$

On trouve de même, en désignant par α', β', γ' des constantes

$$\frac{\partial \psi}{\partial x}=\lambda(\alpha' y+\beta'), \quad \frac{\partial \psi}{\partial y}=-\lambda(\alpha' x+\gamma'),$$
$$2\lambda\alpha'+\frac{\partial \lambda}{\partial y}(\alpha' y+\beta')+\frac{\partial \lambda}{\partial x}(\alpha' x+\gamma')=0. \tag{8'}$$

On a ainsi deux équations du premier degré en $\frac{\partial \lambda}{\partial x}$, $\frac{\partial \lambda}{\partial y}$ dans lesquelles le déterminant des coefficients des inconnues

$$(\alpha x+\gamma)(\alpha' y+\beta')-(\alpha' x+\gamma')(\alpha y+\beta)$$

n'est pas nul, car s'il l'était, le determinant fonctionnel

$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y}-\frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x}$$

serait *nul.* Si l'on posē, pour abréger,

$$\alpha\beta'-\beta\alpha'=a'', \quad \gamma\alpha'-\alpha\gamma'=b'', \quad \gamma\beta'-\beta\gamma'=c''$$

l'on a, en résolvant les deux équations du premier degré (8) et (8'),

$$\frac{1}{\lambda}\frac{\partial \lambda}{\partial x}=\frac{-2a''}{a''x+b''y+c''}, \quad \frac{1}{\lambda}\frac{\partial \lambda}{\partial y}=\frac{-2b''}{a''x+b''y+c''},$$

d'où en intégrant

$$\log \lambda=-2\log(a''x+b''y+c'')-\log k,$$

$$\lambda=\frac{1}{k(a''x+b''y+c'')^2},$$

k désignant une constante arbitraire. Ayant ainsi determiné λ, on calculera les fonctions ϕ et ψ à l'aide des relations

$$\frac{\partial \phi}{\partial x}=\lambda(\alpha y+\beta), \quad \frac{\partial \phi}{\partial y}=-\lambda(\alpha x+\gamma),$$

$$\frac{\partial \psi}{\partial x}=\lambda(\alpha' y+\beta'), \quad \frac{\partial \psi}{\partial y}=-\lambda(\alpha' x+\gamma'),$$

qui donnent pour ϕ et ψ des expressions de la forme

$$\phi=\frac{ax+by+c}{a''x+b''y+c''}, \quad \psi=\frac{a'x+b'y+c'}{a''x+b''y+c''},$$

a, b, c, a', b', c' désignant des constantes. On retrouve ainsi la transformation homographique définie par les relations (2) et (3); et l'on voit que *cette transformation est la seule pour laquelle la force F_1 ne dépende pas de la vitesse, quelle que soit la loi de la force F en fonction de la position du mobile* (x, y).

4). Si la force F, tout en ne dépendant que de la position de son point d'application, était assujettie à des *conditions particulières* il pourrait exister d'autres transformations de la forme précédente

$$x_1=\phi(x, y), \quad y_1=\psi(x, y), \quad dt_1=\lambda(x, y)\,dt$$

pour lesquelles F_1 dépendrait seulement de la position du mobile (x_1, y_1).

Cela arriverait, par exemple, dans le cas particulier où la force F serait centrale, et dans le cas plus particulier encore où le mouvement serait rectiligne. Prenons ce dernier cas et supposons que le mouvement se passe sur l'axe Ox; l'équation du mouvement sera

$$m\frac{d^2x}{dt^2}=X,$$

X dépendant de x seulement. Faisons

$$x_1 = \phi(x), \quad dt_1 = \phi'(x)\,dt,$$

l'équation prendra la forme

$$m\frac{d^2x_1}{dt_1^2} = \frac{X}{\phi'(x)}$$

où le second membre peut s'exprimer en fonction de x_1 seulement. On a ainsi une transformation, avec une fonction arbitraire $\phi(x)$, transformant le mouvement rectiligne d'un point, sous l'action d'une force qui ne dépend que de la position du mobile, en un autre mouvement rectiligne de même nature.

5). Si nous revenons au cas général nous verrons sans peine que les considérations que nous avons développées peuvent être étendues au mouvement d'un point dans l'espace et même au mouvement de plusieurs points sous l'action de forces ne dépendant que de la position des mobiles.

En effet les équations du mouvement, dans un problême de ce genre, sont de la forme

$$\frac{d^2x_1}{dt^2} = X_1, \quad \frac{d^2x_2}{dt^2} = X_2, \ldots . \frac{d^2x_n}{dt^2} = X_n, \tag{9}$$

$X_1, X_2, \ldots . X_n$ étant des fonctions données de $x_1, x_2, \ldots . x_n$.

Faisons la transformation homographique générale

$$y_i = \frac{a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + a_{i,\,n+1}}{\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n + \alpha_{n+1}} = \frac{P_i}{Q}, \quad (i = 1, 2, \ldots . n) \tag{10}$$

$a_{i,1}, a_{i,2} \ldots . \alpha_1, \alpha_2 \ldots .$ étant des constantes, et $y_1, y_2 \ldots .$ de nouvelles fonctions que l'on substitue à $x_1, x_2 \ldots .$; faisons en même temps le changement de variable indépendante

$$dt_1 = \frac{dt}{Q^2}.$$

Nous aurons alors

$$\frac{dy_i}{dt_1} = \frac{dy_i}{dt}\cdot\frac{dt}{dt_1} = Q\frac{dP_i}{dt} - P_i\frac{dQ}{dt};$$

puis

$$\frac{d^2y_i}{dt_1^2} = Q^2\frac{d}{dt}\left(Q\frac{dP_i}{dt} - P_i\frac{dQ}{dt}\right) = Q^3\left(Q\frac{d^2P_i}{dt^2} - P_i\frac{d^2Q}{dt^2}\right). \tag{11}$$

Comme Q et P_i sont linéaires en $x_1, x_2 \ldots . x_n$, le second membre de cette relation contient les quantités $x_1, x_2 \ldots . x_n$ et leurs dérivées secondes par rapport à t, mais pas leurs dérivées premières. On pourra donc, à l'aide des équations (9) et (10), exprimer le second membre de cette relation (11) en fonction des seules

quantités $y_1, y_2 \ldots . y_n$. Les équations (9) du mouvement sont ainsi transformées en d'autres de même forme.

Cette transformation homographique est d'ailleurs la seule transformation de la forme

$$y_i = \phi_i(x_1, x_2, \ldots . x_n) \qquad (i = 1, 2, \ldots . n)$$
$$dt_1 = \lambda(x_1, x_2, \ldots . x_n)\, dt,$$

qui transforme les équations (9) où $X_1, X_2 \ldots . X_n$ sont des fonctions quelconques de $x_1, x_2, \ldots . x_n$ en d'autres de même nature relativement aux quantités $y_1, y_2, \ldots . y_n$ et à la variable indépendante t_1.

6). On pourrait généraliser les résultats précédents de la façon suivante qui nous a été signalée par M. Goursat.

Soient les équations de Lagrange

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i,$$
$$q_i' = \frac{dq_i}{dt}, \qquad (i = 1, 2, \ldots . k)$$

où T est une forme quadratique de $q_1', q_2', \ldots . q_k'$ avec des coefficients fonctions de $q_1, q_2, \ldots . q_k$ et où $Q_1, Q_2, \ldots . Q_k$ dépendent seulement de $q_1, q_2, \ldots . q_k$, trouver les transformations de la forme

$$r_i = f_i(q_1, q_2, \ldots . q_k), \qquad (i = 1, 2, \ldots . k)$$
$$dt_1 = \lambda(q_1, q_2, \ldots . q_k)\, dt$$

qui transforment ces équations en d'autres de la forme

$$\frac{d}{dt_1}\left(\frac{\partial S}{\partial r_i'}\right) - \frac{\partial S}{\partial r_i} = R_i,$$
$$r_i' = \frac{dr_i}{dt_1}, \qquad (i = 1, 2, \ldots . k)$$

où S désigne une forme quadratique de $r_1', r_2', \ldots . r_k'$ avec des coefficients fonctions de $r_1, r_2, \ldots r_k$ et où $R_1, R_2, \ldots R_k$ dépendent seulement de $r_1, r_2, \ldots r_k$. Un cas particulièrement intéressant serait celui où la forme S se déduirait de T par le simple changement de

$$q_1, q_2, \ldots . q_k;\ q_1', q_2', \ldots . q_k'$$

en

$$r_1, r_2, \ldots . r_k;\ r_1', r_2', \ldots . r_k'.$$

Il est à présumer que les transformations que l'on trouvera sont analogues à celles qui, dans le cas du mouvement d'un point sur une surface, $(k = 2)$, conservent les lignes géodésiques.

Systems of Ternariants that are Algebraically Complete.

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(Continued from page 60.)

PART III.

APPLICATIONS TO BIPARTITE QUANTICS.

59. The theory given in Part I holds alike for unipartite and for bipartite quantics: the difference in details arises through the difference of the literal operators. If in the quantic, symbolically represented by $a_x^m u_\alpha^\mu$, the coefficient of $x_1^r x_2^s x_3^t\, u_1^\rho u_2^\sigma u_3^\tau$ be

$$\frac{m!}{r!\,s!\,t!}\,\frac{\mu!}{\rho!\,\sigma!\,\tau!}\,a_{r,s,t,\rho,\sigma,\tau},$$

(with the conditions $r+s+t=m$, $\rho+\sigma+\tau=\mu$), then the six operators similar to those of §1 are*

$$\left.\begin{aligned}
D_3 &= \Sigma\,(ra_{r-1,s+1,t,\rho,\sigma,\tau}-\sigma a_{r,s,t,\rho+1,\sigma-1,\tau})\,\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}\\
D_1 &= \Sigma\,(sa_{r,s-1,t+1,\rho,\sigma,\tau}-\tau a_{r,s,t,\rho,\sigma+1,\tau-1})\,\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}\\
D_2 &= \Sigma\,(ta_{r+1,s,t-1,\rho,\sigma,\tau}-\rho a_{r,s,t,\rho-1,\sigma,\tau+1})\,\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}\\
D_4 &= \Sigma\,(sa_{r+1,s-1,t,\rho,\sigma,\tau}-\rho a_{r,s,t,\rho-1,\sigma+1,\tau})\,\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}\\
D_6 &= \Sigma\,(ta_{r,s+1,t-1,\rho,\sigma,\tau}-\sigma a_{r,s,t,\rho,\sigma-1,\tau+1}\,\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}\\
D_5 &= \Sigma\,(ra_{r-1,s,t+1,\rho,\sigma,\tau}-\tau a_{r,s,t,\rho+1,\sigma,\tau-1})\,\frac{\partial}{\partial a_{r,s,t,\rho,\sigma,\tau}}
\end{aligned}\right\} \qquad (1').$$

* See the memoir there cited, p. 42, where the signs + within the bracket should be changed to —.

I.—*The Lineo-Linear Quantic.*

60. In the symbolical form this is $a_x u_a$;* the explicit form we shall take to be

$$U = (a_1x_1 + h_1x_2 + g_1x_3)\,u_1 + (a_2x_1 + h_2x_2 + g_2x_3)\,u_2 + (a_3x_1 + h_3x_2 + g_3x_3)\,u_3;$$

the sequence of literal coefficients and the arrangement of numerical subscripts

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* The principal memoirs dealing with the theory of bilinear forms are given in the following list:

Weierstrass (1858). "Ueber ein die homogenen Functionen zweiten Grades betreffendes Theorem." Berl. Monatsb., 1858, pp. 207–220.

Kronecker (1866). "Ueber bilineare Formen." Berl. Monatsb., 1866, pp. 597–612; reprinted in Crelle, t. LXVIII (1868), pp. 273–285.

Weierstrass (1868). "Zur Theorie der bilinearen und quadratischen Formen." Berl. Monatsb., 1868, pp. 310–338.

Kronecker (1868). Bemerkungen zu vorstehendem Vortrag. Ib., pp. 339–346.

Christoffel (1868). "Theorie der bilinearen Functionen." Crelle, t. LXVIII, pp. 253–272.

Clebsch und Gordan (1869). "Ueber biternäre Formen mit contragredienten Variabeln." Math. Ann., t. i, pp. 359–400; specially pp. 371–400.

Beltrami (1873). "Sulle funzioni bilineari." Batt. Giorn., t. XI, pp. 98–106.

Jordan (1873). "Sur les polynômes bilinéaires." Comptes Rendus, t. LXXVII, pp. 1487–1491; Liouville, 2e Sér., t. XIX, pp. 35–54.

Jordan (1874). "Sur la réduction des formes bilinéaires." Comptes Rendus, t. LXXVIII, pp. 614–617.
"Sur les systèmes de formes quadratiques." Comptes Rendus, t. LXXVIII, pp. 1763–1767.

Kronecker (1874). "Ueber Schaaren von quadratischen Formen." Berl. Monatsb., 1874, pp. 59–76.
Nachtrag zu diesem Aufsatze. Ib., pp. 149–156.
"Ueber Schaaren von quadratischen und bilinearen Formen." Ib., pp. 206–232.
"Ueber die congruenten Transformationen der bilinearen Formen." Ib., pp. 397–447.

Darboux (1874). "Mémoire sur la théorie algébrique des formes quadratiques." Liouville, 2e Sér., t. XIX, pp. 347–396.

Jordan (1874). "Mémoire sur la réduction et la transformation des systèmes quadratiques." Ib., pp. 397–422.

Frobenius (1878). "Ueber lineare Substitutionen und bilineare Formen." Crelle, t. LXXXIV, pp. 1–63.

Bachmann (1873). "Untersuchungen über quadratischen Formen." Crelle, t. LXXVI, pp. 331–341.

Hermite (1874). Extrait d'un lettre à M. Borchardt. Crelle, t. LXXVIII, pp. 325–328.

Cayley. "Sur la transformation d'une fonction quadratique en elle-même par des substitutions linéaires." Crelle, t. L, pp. 288–299.

Stickelberger (1879). "Ueber Schaaren von bilinearen und quadratischen Formen." Crelle, t. LXXXVI, pp. 20–43.

Frobenius (1879). "Ueber die schiefe Invarianten einer bilinearen oder quadratischen Form." Ib., pp. 44–71.

Cayley (1858). "A memoir on the automorphic linear transformation of a bipartite quadric function." Phil. Trans., 1858, pp. 39–46.

Other references will be found in these memoirs which deal very largely with the transformation and canonization of the forms; the memoirs of Christoffel, Clebsch and Gordan, and the last by Frobenius, deal with the concomitants.

will be seen to harmonize with those in later examples. The six operators in (1′) are

$$D_3 = h_1 \frac{\partial}{\partial a_1} + h_2 \frac{\partial}{\partial a_2} + h_3 \frac{\partial}{\partial a_3} - a_1 \frac{\partial}{\partial a_2} - h_1 \frac{\partial}{\partial h_2} - g_1 \frac{\partial}{\partial g_2},$$
$$D_1 = g_1 \frac{\partial}{\partial h_1} + g_2 \frac{\partial}{\partial h_2} + g_3 \frac{\partial}{\partial h_3} - a_2 \frac{\partial}{\partial a_3} - h_2 \frac{\partial}{\partial h_3} - g_2 \frac{\partial}{\partial g_3},$$
$$D_2 = a_1 \frac{\partial}{\partial g_1} + a_2 \frac{\partial}{\partial g_2} + a_3 \frac{\partial}{\partial g_3} - a_3 \frac{\partial}{\partial a_1} - h_3 \frac{\partial}{\partial h_1} - g_3 \frac{\partial}{\partial g_1},$$
$$D_4 = a_1 \frac{\partial}{\partial h_1} + a_2 \frac{\partial}{\partial h_2} + a_3 \frac{\partial}{\partial h_3} - a_2 \frac{\partial}{\partial a_1} - h_2 \frac{\partial}{\partial h_1} - g_2 \frac{\partial}{\partial g_1},$$
$$D_6 = h_1 \frac{\partial}{\partial g_1} + h_2 \frac{\partial}{\partial g_2} + h_3 \frac{\partial}{\partial g_3} - a_3 \frac{\partial}{\partial a_2} - h_3 \frac{\partial}{\partial h_2} - g_3 \frac{\partial}{\partial g_2},$$
$$D_5 = g_1 \frac{\partial}{\partial a_1} + g_2 \frac{\partial}{\partial a_2} + g_3 \frac{\partial}{\partial a_3} - a_1 \frac{\partial}{\partial a_3} - h_1 \frac{\partial}{\partial h_3} - g_1 \frac{\partial}{\partial g_3}.$$

From the general theory it follows that every concomitant has its leading coefficient a common solution of $D_1 = 0$ and $D_6 = 0$. Proceeding therefore to obtain these solutions, we first consider the set of equations subsidiary to $D_1 = 0$ given by

$$\frac{da_1}{0} = \frac{dg_1}{0} = \frac{da_2}{0} = \frac{dg_2}{0} = \frac{dh_1}{g_1} = \frac{dh_2}{g_2} = \frac{dh_3}{g_3 - h_2} = \frac{da_3}{-a_2} = \frac{dg_3}{-g_2},$$

eight in all; eight independent integrals of them will therefore be required. We have

$$\theta_1 = a_1,$$
$$\theta_2 = a_2,$$
$$\theta_3 = g_1,$$
$$\theta_4 = g_2;$$

and we apply the process of §35. Changing D_6 to Δ, we have

$$\Delta\theta_1 = 0,$$

so that θ_1 is a solution common to the two equations. Now for the 'variable of reference' we might take either a_2 or g_1; taking the latter, we have θ_3 as the connecting variable of the modified equations in Δ. We have

$$\Delta\theta_3 = h_1, \quad \Delta\theta_2 = -a_3,$$

so that
$$\theta_3\Delta\theta_2 - \theta_2\Delta\theta_3 = -(a_3 g_1 + a_2 h_1) = -\theta_5,$$

this quantity θ_5 being easily verified to be another solution of the subsidiary set. And then
$$\Delta\theta_5 = 0,$$

so that θ_5 is a solution common to $D_1 = 0$ and $D_6 = 0$.

Next we have $$\Delta\theta_4 = h_2 - g_3,$$
so that $$\theta_3\Delta\theta_4 - 2\theta_4\Delta\theta_3 = g_1(h_2 - g_3) - 2h_1g_2 = 2\theta_6,$$
where θ_6 is easily verified to be another solution of the subsidiary set. Next we have $\Delta\theta_6 = -\frac{1}{2}h_1(h_2 - g_3) - h_3g_1$, and therefore

$$\theta_3\Delta\theta_6 - \theta_6\Delta\theta_3 = -h_3g_1^2 + g_1h_1(g_3 - h_2) + g_2h_1^2 = \theta_7,$$

where θ_7 is a solution of the subsidiary set. And then

$$\Delta\theta_7 = 0,$$

so that θ_7 is a solution common to $D_1 = 0$ and $D_6 = 0$.

We now have seven integrals of the subsidiary set; the remaining one necessary we may take to be
$$\theta_8 = h_2 + g_3;$$
and we have $$\Delta\theta_8 = 0,$$
so that θ_8 is a solution common to the two characteristic equations.

It is easy to see that the eight integrals thus chosen are independent of one another.

61. Four common solutions of the two characteristic equations are $\theta_1, \theta_5, \theta_7, \theta_8$; for the remaining two that (§35) are necessary, we have

$$\theta_3^2\Delta(\theta_2 \div \theta_3) = -\theta_5;$$
$$\theta_3^2\Delta(\theta_4 \div \theta_3^2) = 2\theta_6 \div \theta_3,$$
$$\theta_3^2\Delta(\theta_6 \div \theta_3) = \theta_7,$$

of which two solutions are

$$\phi_1 = \frac{\theta_6^2 - \theta_4\theta_7}{\theta_3^2} = \frac{1}{4}\theta_8^2 + g_2h_3 - g_3h_2,$$

and $$\phi_2 = \frac{\theta_5\theta_6 + \theta_2\theta_7}{\theta_3} = -h_3a_2g_1 + \frac{1}{2}(g_3 - h_2)(a_2h_1 - a_3g_1) - g_2a_3h_1.$$

There are thus six common solutions, the values of which in terms of the coefficients of the quantic are

$$\theta_1 = v_0 = a_1,$$
$$\theta_5 = v_1 = (a_3,\ a_2 \between g_1,\ h_1),$$
$$\theta_7 = v_2 = (-h_3,\ g_3 - h_2,\ g_2 \between g_1,\ h_1)^2,$$
$$\theta_8 = g_3 + h_2;$$

instead of ϕ_1, which is practically the discriminant of v_2 considered as a binary form, we take

$$\vartheta_2 = \phi_1 - \frac{1}{4}\theta_8^2 = g_2 h_3 - g_3 h_2;$$

and instead of ϕ_3, which is the Jacobian of v_1 and v_2 considered as binary forms in g_1 and h_1 as variables, we take

$$\begin{aligned}\vartheta_3 &= \phi_3 + \frac{1}{2}\theta_5\theta_8 \\ &= a_2(g_3 h_1 - g_1 h_3) + a_3(g_1 h_2 - g_2 h_1).\end{aligned}$$

To obtain the order and the class to be associated with each of these quantities as leading coefficients of a concomitant, we use (I) and (II) and easily find,

$$\begin{aligned}\Theta_1 &= \theta_1 x_1 u_1 + \dots, \\ \Theta_5 &= \theta_5 x_1^2 u_1^2 + \dots, \\ \Theta_7 &= \theta_7 u_1^3 \quad + \dots, \\ \Theta_8 &= \theta_8 x_1 u_1 + \dots, \\ \Theta_2 &= \vartheta_2 x_1 u_1 + \dots, \\ \Theta_3 &= \vartheta_3 x_1^2 u_1^2 + \dots\end{aligned}$$

Further, we at once have

$$\theta_1 + \theta_8 = I_1,$$

an invariant, and

$$u_x I_1 = \Theta_1 + \Theta_8;$$

and also

$$\theta_1\vartheta_2 + \vartheta_3 = \begin{vmatrix} a_1, & a_2, & a_3 \\ g_1, & g_2, & g_3 \\ h_1, & h_2, & h_3 \end{vmatrix} = I_3,$$

an invariant, and

$$u_x^2 I_3 = \Theta_1\Theta_2 + \Theta_3.$$

Another invariant, of the second degree, is given in §63.

It follows from the general theory that *every concomitant of the lineo-linear ternary quantic can be expressed in terms of the six independent concomitants* $\Theta_1, \Theta_5, \Theta_7, \Theta_8, \Theta_2, \Theta_3$.

62. But if, instead of taking in §60 the quantity θ_8 as the variable of reference, we take θ_2 as that variable, we are led to the following system of independent solutions common to the two characteristic equations:

$$\begin{aligned}\theta_1 &= a_1, \\ \theta_5 &= (h_1, g_1 \between a_2, a_3), \\ \theta_8 &= g_3 + h_2, \\ \vartheta_2 &= g_2 h_3 - g_3 h_2, \\ \vartheta_3 &= a_2(g_3 h_1 - g_1 h_3) + a_3(g_1 h_2 - g_2 h_1), \\ \phi_7 &= (-h_3, h_2 - g_3, g_2 \between a_2, a_3)^2,\end{aligned}$$

the first five of which are the same as before and necessarily lead to the same concomitants, while the sixth leads to the concomitant

$$\Phi_7 = \phi_7 x_1^3 + \ldots .$$

Hence (§61, fin.) Φ_7 must be expressible in terms of $\Theta_1, \Theta_5, \Theta_7, \Theta_8, \Theta_2, \Theta_3$.

63. Now Clebsch and Gordan have already* given the complete system of asyzygetic concomitants of the bilinear ternary quantic; they are, in addition to u_x,

Symbol (*C.* and *G.*)	Symbolic Form.	Evaluate Form.
f	$= a_x u_\alpha$	$= a_1 x_1 u_1 + \ldots ,$
i	$= a_\alpha$	$= a_1 + h_2 + g_3,$
i_1	$= a_\beta b_\alpha$	$= a_1^2 + h_2^2 + g_3^2 + 2a_2 h_1 + 2a_3 g_1 + 2g_2 h_3,$
f_1	$= a_x u_\beta b_\alpha$	$= (a_1^2 + a_2 h_1 + a_3 g_1) x_1 u_1 + \ldots ,$
i_2	$= a_\beta b_\gamma c_\alpha$	$= \frac{3}{2} i i_1 - \frac{1}{2} i_1^3 + \begin{vmatrix} a_1, & a_2, & a_3 \\ h_1, & h_2, & h_3 \\ g_1, & g_2, & g_3 \end{vmatrix},$
ϕ	$= a_x c_x b_\alpha (\beta\gamma x)$	$= \{- a_2^2 h_3 + a_2 a_3 (h_2 - g_3) + g_2 a_3^2\} x_1^3 + \ldots ,$
ψ	$= u_\beta u_\gamma b_\alpha (acu)$	$= \{g_1^2 h_3 + g_1 h_1 (h_2 - g_3) - h_1^2 g_2\} u_1^3 + \ldots .$

From the theorem that the system $\Theta_1, \Theta_5, \Theta_7, \Theta_8, \Theta_2, \Theta_3$ is algebraically complete, it follows that each one of this set can be expressed in terms of members of that system; and, in fact, it is easy to prove the relations:

$$\begin{aligned}
f &= \Theta_1, \\
u_x i &= \Theta_1 + \Theta_3, \\
u_x^2 i_1 &= \Theta_1^2 + \Theta_3^2 + 2\Theta_5 + 2u_x\Theta_2, \\
u_x f_1 &= \Theta_1^2 + \Theta_5, \\
u_x^3 i_2 &= u_x^3 \left(\tfrac{3}{2} i i_1 - \tfrac{1}{2} i^3\right) - \Theta_1\Theta_2 - \Theta_8, \\
\phi &= \Phi_7, \\
\psi &= -\Theta_7.
\end{aligned}$$

And the expression of Φ_7 in terms of the system of six concomitants Θ is easily shown to be given by

$$u_x^2 \Theta_7 \Phi_7 = u_x \Theta_8^2 - \Theta_3 \Theta_5 \Theta_8 - \Theta_2 \Theta_5^2 .$$

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* See pp. 372 et seq. of their memoir, quoted in the note to §60.

This equation is, after the foregoing relations, equivalent to an equation among Clebsch and Gordan's forms; the verification of this result is easily obtainable from the canonical forms of the concomitants (p. 386 l. c.).

64. If, in the two equations of §60 which determine leading coefficients, we take

$$g_3 - h_2 = 2k, \quad g_3 + h_2 = \theta_3,$$

we find

$$D_1 = g_1 \frac{\partial}{\partial h_1} - a_2 \frac{\partial}{\partial a_3} - g_2 \frac{\partial}{\partial k} + 2k \frac{\partial}{\partial h_3} = 0,$$

$$D_6 = h_1 \frac{\partial}{\partial g_1} - a_3 \frac{\partial}{\partial a_2} + h_3 \frac{\partial}{\partial k} - 2k \frac{\partial}{\partial g_2} = 0.$$

When these are written in the forms

$$g_1 \frac{\partial}{\partial h_1} - a_2 \frac{\partial}{\partial a_3} = \quad g_2 \frac{\partial}{\partial k} - 2k \frac{\partial}{\partial h_3},$$

$$h_1 \frac{\partial}{\partial g_1} - a_3 \frac{\partial}{\partial a_2} = - h_3 \frac{\partial}{\partial k} + 2k \frac{\partial}{\partial g_2},$$

they are the differential equations of *the concomitants of a binary quadratic in literal coefficients* $-h_3, k, g_2$, *the variables in the concomitants being* g_1 and h_1, a_2 and $-a_3$.

To this system of solutions must be added a_1 and θ_3, neither of which enters into the transformed equations and both of which are therefore solutions.

This inference is verified immediately on a reference to the system of solutions obtained, the solution θ_5 being the determinant of the variables.

Similar inferences in the succeeding cases may be derived after the characteristic equations have been transformed by similar substitutions; but the inferences will be left unstated, because obvious from the respective systems of solutions, and the equations will not be transformed, the requisite substitutions being indicated sufficiently by the respective systems of solutions.

II.—*A System of Two Lineo-Linear Quantics.*

65. They may be taken in the forms

$$U = (a_1x_1 + h_1x_2 + g_1x_3)\,u_1 + (a_2x_1 + h_2x_2 + g_2x_3)\,u_2 + (a_3x_1 + h_3x_2 + g_3x_3)\,u_3,$$

$$U' = (a_1'x_1 + h_1'x_2 + g_1'x_3)\,u_1 + (a_2'x_1 + h_2'x_2 + g_2'x_3)\,u_2 + (a_3'x_1 + h_3'x_2 + g_3'x_3)\,u_3,$$

and the literal operators are now of the form $D_\lambda + D_\lambda'$ instead of D_λ. We must therefore find the simultaneous independent solutions of

$$D_1 + D_1' = 0, \quad \Delta = D_6 + D_6' = 0.$$

From the 17 equations, which are subsidiary to the solution of the former equation, it at once appears that there are eight integrals of the form

$$\left.\begin{array}{llll}\theta_1 = a_1, & \theta_2 = a_2, & \theta_3 = g_1, & \theta_4 = g_2 \\ \theta_1' = a_1', & \theta_2' = a_2', & \theta_3' = g_1', & \theta_4' = g_2'\end{array}\right\};$$

and these integrals are used for the deduction of further integrals by the usual method hitherto adopted, and also for the modification of the Δ-equation. Also, as in the case of a single lineo-linear quantic it was possible to take either θ_3 or θ_2 as variable of reference, it is now possible to take any one of the four θ_2, θ_3, θ_2', θ_3' as variable of reference.

As in the earlier case of the system of three quadratics (53–58), I merely give some important stages of the work and the results.

Since there are eighteen coefficients, it follows (§18) that *all the simultaneous solutions can be expressed in terms of fifteen independent simultaneous solutions.*

66. In addition to the preceding eight quantities θ, which are solutions of $D_1 + D_1' = 0$ and among which θ_1 and θ_1' are also solutions of $\Delta = 0$, the following quantities—all being solutions of $D_1 + D_1' = 0$—arise in the formation of the modified Δ-equations:

$$\left.\begin{array}{l}\theta_6 = (h_2 - g_3)\, g_1 - 2g_2 h_1 \\ \theta_6' = (h_2' - g_3')\, g_1' - 2g_2' h_1'\end{array}\right\}, \quad \left.\begin{array}{l}\psi_6 = (h_2' - g_3')\, g_1 - 2g_2' h_1 \\ \psi_6' = (h_2 - g_3)\, g_1' - 2g_2 h_1'\end{array}\right\};$$

$$\left.\begin{array}{l}\chi_6 = (h_2 - g_3)\, a_2 + 2g_2 a_3 \\ \chi_6' = (h_2' - g_3')\, a_2' + 2g_2' a_3'\end{array}\right\}, \quad \left.\begin{array}{l}\phi_6 = (h_2' - g_3')\, a_2 + 2g_2' a_3 \\ \phi_6' = (h_2 - g_3)\, a_2' + 2g_2 a_3'\end{array}\right\}.$$

And in the modified Δ-equations in the fourteen quantities, so far obtained yet not simultaneous solutions—viz. θ_2, θ_3, θ_4, θ_2', θ_3', θ_4' and the eight just given—the following further quantities arise, all of them being simultaneous solutions of the two characteristic equations:

$$\left.\begin{array}{l}\theta_5 = a_3 g_1 + a_2 h_1 \\ \theta_5' = a_3' g_1' + a_2' h_1'\end{array}\right\}, \quad \left.\begin{array}{l}\psi_5 = a_3' g_1 + a_2' h_1 \\ \psi_5' = a_3 g_1' + a_2 h_1'\end{array}\right\};$$

$$\phi = g_1 h_1' - g_1' h_1; \qquad \psi = a_2' a_3 - a_3' a_2;$$

$$\left.\begin{array}{l}\theta_7 = (A,\, B,\, C \between g_1,\, -h_1)^2 \\ \lambda_7 = (A,\, B,\, C \between g_1,\, -h_1 \between g_1',\, -h_1') \\ \psi_7' = (A,\, B,\, C \between g_1',\, -h_1')^2\end{array}\right\}; \quad \left.\begin{array}{l}\psi_7 = (A',\, B',\, C' \between g_1,\, -h_1)^2 \\ \lambda_7' = (A',\, B',\, C' \between g_1,\, -h_1 \between g_1',\, -h_1') \\ \theta_7' = (A',\, B',\, C' \between g_1',\, -h_1')^2\end{array}\right\};$$

$$\left.\begin{array}{l}\chi_7 = (A,\, B,\, C \between a_2,\, a_3)^2 \\ \rho_7 = (A,\, B,\, C \between a_2,\, a_3 \between a_2',\, a_3') \\ \phi_7' = (A,\, B,\, C \between a_2',\, a_3')^2\end{array}\right\}; \quad \left.\begin{array}{l}\phi_7 = (A',\, B',\, C' \between a_2,\, a_3)^2 \\ \rho_7' = (A',\, B',\, C' \between a_2,\, a_3 \between a_2',\, a_3') \\ \chi_7' = (A',\, B',\, C' \between a_2',\, a_3')^2\end{array}\right\};$$

$$\left.\begin{aligned}
\phi_2 &= (A, B, C)(g_1, -h_1)(a_2, a_3) \\
v_2 &= (A, B, C)(g_1, -h_1)(a_2', a_3') \\
\omega_2 &= (A, B, C)(g_1', -h_1')(a_2, a_3) \\
\chi_2' &= (A, B, C)(g_1', -h_1')(a_2', a_3')
\end{aligned}\right\} ; \quad
\left.\begin{aligned}
\chi_2 &= (A', B', C')(g_1, -h_1)(a_2, a_3) \\
\omega_2' &= (A', B', C')(g_1, -h_1)(a_2', a_3') \\
v_2' &= (A', B', C')(g_1', -h_1')(a_2, a_3) \\
\phi_2' &= (A', B', C')(g_1', -h_1')(a_2', a_3')
\end{aligned}\right\} ;$$

the quantities A, B, C and A', B', C' being

$$\begin{aligned}
A &= -2h_3', \quad B = h_2 - g_3, \quad C = 2g_2, \\
A' &= -2h_3', \quad B' = h_2' - g_3', \quad C' = 2g_2'.
\end{aligned}$$

The following are the sets of modified Δ-equations in each of the four possible variables of reference; the first seven equations of each set are the independent equations in that set, but their aggregate in any one set is rendered (as in §47) complete in form by the introduction of the subsidiary quantities which occur in all the other sets. The equations are:

$\theta_3^2\Delta = \nabla_3$	$\theta_2^2\Delta = \nabla_2$	$\theta_3'^2\Delta = \nabla_3'$	$\theta_2^2\Delta = \nabla_2'$
$\nabla_3\epsilon = -\theta_5$	$\nabla_2\epsilon' = \theta_5$	$\nabla_3'\mu = -\theta_5'$	$\nabla_2'\mu' = \theta_5'$
$\nabla_3 p = q$	$\nabla_2 r = s$	$\nabla_3' p' = q'$	$\nabla_2' r' = s'$
$\nabla_3 q = \theta_7$	$\nabla_2 s = \chi_7$	$\nabla_3' q' = \theta_7'$	$\nabla_2' s' = \chi_7'$
$\nabla_3\gamma = \phi$	$\nabla_2\iota = \psi$	$\nabla_3'\delta' = -\psi_5'$	$\nabla_2'\iota' = -\psi$
$\nabla_3\alpha = -\psi_5$	$\nabla_2\delta = \psi_5'$	$\nabla_3'\gamma' = -\phi$	$\nabla_2'\alpha' = \psi_5$
$\nabla_3\pi = \varkappa$	$\nabla_2\rho = \phi_6$	$\nabla_3'\pi' = \varkappa'$	$\nabla_2'\rho' = \sigma'$
$\nabla_3\varkappa = \psi_7$	$\nabla_2\sigma = \phi_7$	$\nabla_3'\varkappa' = \psi_7'$	$\nabla_2'\sigma' = \phi_7'$
$\nabla_3 t = \phi_2$	$\nabla_2\xi = \phi_2$	$\nabla_3' u' = \lambda_7$	$\nabla_2'\zeta' = u_2$
$\nabla_3 l = \chi_2$	$\nabla_2\eta = \chi_2$	$\nabla_3' x' = \lambda_7'$	$\nabla_2'\omega' = \omega_2'$
$\nabla_3 u = \lambda_7'$	$\nabla_2\zeta = v_2'$	$\nabla_3'\tau' = \omega_2$	$\nabla_2' g' = \rho_7$
$\nabla_3 x = \lambda_7$	$\nabla_2\omega = \omega_2$	$\nabla_3'\lambda' = v_2'$	$\nabla_2' f' = \rho_7'$
$\nabla_3\tau = \omega_2'$	$\nabla_2 g = \rho_7'$	$\nabla_3' t' = \phi_2'$	$\nabla_2'\xi' = \phi_2'$
$\nabla_3\lambda = v_2$	$\nabla_2 f = \rho_7$	$\nabla_3' l' = \chi_2'$	$\nabla_2'\eta' = \chi_2'$

where the various quantities are defined by the relations

$$\left.\begin{aligned}\theta_2 &= \theta_3\varepsilon \quad = \theta_3'\delta' = \theta_2'\iota' \\ \theta_3 &= \theta_2\varepsilon' = \theta_3'\gamma' = \theta_2'\alpha'\end{aligned}\right\}, \quad \left.\begin{aligned}\theta_2' &= \theta_3\alpha = \theta_2\iota = \theta_3'\mu \\ \theta_3' &= \theta_3\gamma = \theta_2\delta \quad = \theta_2'\mu'\end{aligned}\right\},$$

$$\left.\begin{aligned}\theta_4 &= \theta_3^2 p = \theta_2^2 r = \theta_3'^2\pi' = \theta_2'^2\rho' \\ \theta_4' &= \theta_3^2\pi = \theta_2^2\rho = \theta_3'^2 p' = \theta_2'^2 r'\end{aligned}\right\},$$

$$\left.\begin{aligned}\theta_6 &= \theta_3 q = \theta_2\xi = \theta_3' u' = \theta_2'\zeta' \\ \theta_6' &= \theta_3 u = \theta_2\zeta = \theta_3' q' = \theta_2'\xi'\end{aligned}\right\}, \quad \left.\begin{aligned}\phi_6 &= \theta_3 l = \theta_2\sigma = \theta_3'\lambda' = \theta_2' f' \\ \phi_6' &= \theta_3\lambda = \theta_2 f = \theta_3' l' = \theta_2'\sigma'\end{aligned}\right\},$$

$$\left.\begin{aligned}\psi_6 &= \theta_3\varkappa = \theta_2\eta = \theta_3' v' = \theta_2'\omega' \\ \psi_6' &= \theta_3 x = \theta_2\omega = \theta_3'\varkappa' = \theta_2'\eta'\end{aligned}\right\}, \quad \left.\begin{aligned}\chi_6 &= \theta_3 t = \theta_2 s = \theta_3'\tau' = \theta_2' g' \\ \chi_6' &= \theta_3\tau = \theta_2 g = \theta_3' t' = \theta_2' s'\end{aligned}\right\}.$$

67. The independent simultaneous solutions of the two characteristic equations can be derived from any one of the sets; let us, then, consider the first set, retaining for this purpose only the (first seven) independent equations. Regarding these equations as furnishing one system of common integrals, we have the eight solutions of $D_1 + D_1' = 0$ before given, θ_1, θ_2, θ_3, θ_4 and θ_1', θ_2', θ_3', θ_4'; and those constructed later, viz. θ_5; θ_6, θ_7; ϕ; ψ_5; ψ_6, ψ_7; being fifteen in all. Two more are necessary to make up the requisite number of seventeen solutions of $D_1 + D_1' = 0$; and they (as in §60) may be taken

$$\theta_8 = h_2 + g_3,$$
$$\theta_8' = h_2' + g_3'.$$

Now θ_8 and θ_8' both satisfy $\Delta = 0$, and therefore it appears that of the necessary fifteen simultaneous solutions we already have

$$\theta_1,\ \theta_1';\ \theta_8,\ \theta_8';\ \theta_5,\ \psi_5;\ \theta_7,\ \psi_7;\ \text{and } \phi,$$

so that six independent integrals—the proper number—must be obtained from the modified (∇_3-)equations. Six solutions, algebraically independent of one another and of those already obtained, are:

$$\begin{aligned}\varepsilon\theta_7 + q\theta_5 &= \phi_2, \\ q^2 - 2p\theta_7 &= (h_2 - g_3)^2 + 4h_3 g_2 = \Delta, \\ \varepsilon\phi + \gamma\theta_5 &= \psi_5', \\ \alpha\theta_5 - \varepsilon\psi_5 &= \psi, \\ \varkappa^2 - 2\pi\psi_7 &= (h_2' - g_3')^2 + 4h_3' g_2' = \Delta', \\ \varepsilon\psi_7 + \varkappa\theta_5 &= \chi_3.\end{aligned}$$

Hence every simultaneous solution can be expressed in terms of the 15 ($= 9 + 6$) solutions already obtained.

68. These fifteen solutions are not, however, symmetrical with regard to the two quantics, and they will therefore be replaced by a system which is symmetrical and at the same time is algebraically equivalent to them. Among special relations—a fuller system will be given immediately—we have

$$\theta_5\theta_5' = \psi_5\psi_5' + \psi\phi,$$

so that we may replace ψ by θ_5'; we have

$$\begin{aligned} \theta_7'\psi_7 &= \lambda_7'^2 \quad - \phi^2\Delta', \\ \lambda_7'\theta_5 &= \psi_7\psi_5' - \phi\chi_2, \end{aligned}$$

so that we may replace χ_2 by θ_7'; and the second of these equations we shall use with

$$\lambda_7'\theta_5' = \phi\phi_2' + \theta_7'\psi_5$$

to replace ψ_5 by ϕ_2'; and lastly, we have

$$\begin{aligned} \theta_7\psi_7' &= \lambda_7^2 \quad - \phi^2\Delta, \\ \lambda_7\theta_5 &= \theta_7\psi_5' - \phi\phi_2, \end{aligned}$$

so that we may replace ψ_5' by ψ_7'.

The set is now constituted by

$$\left.\begin{matrix}\theta_1\\ \theta_1'\end{matrix}\right\}, \left.\begin{matrix}\theta_8\\ \theta_8'\end{matrix}\right\}, \left.\begin{matrix}\Delta\\ \Delta'\end{matrix}\right\}, \left.\begin{matrix}\theta_5\\ \theta_5'\end{matrix}\right\}, \left.\begin{matrix}\theta_7\\ \theta_7'\end{matrix}\right\}, \left.\begin{matrix}\phi_2\\ \phi_2'\end{matrix}\right\}, \psi_7, \psi_7', \phi,$$

a system symmetrical with regard to the two quantics; and it follows that *every simultaneous solution can be expressed in terms of this set.*

Two of the set, viz. Δ and Δ', may be simplified in form as in §61, for

$$\begin{aligned} \tfrac{1}{4}(\Delta - \theta_8^2) &= g_2h_3 - g_3h_2 = \vartheta_2, \\ \tfrac{1}{4}(\Delta' - \theta_8'^2) &= g_2'h_3' - g_3'h_2' = \vartheta_2', \end{aligned}$$

which will be taken as leading coefficients when the system of concomitants is established; but for the purpose of expressing dependent solutions it is convenient to retain Δ and Δ'. And as an intermediate quantity we have

$$\Delta_{12} = \varkappa q - p\psi_7 - \pi\theta_7 = (h_2 - g_3)(h_2' - g_3') + 2h_3'g_2 + 2h_3g_2',$$

a simultaneous solution of the two characteristic equations, which also admits of some simplification in form by taking

$$\vartheta_{12} = \frac{1}{2}(\Delta_{12} - \theta_8\theta_8') = g_2h_3' + g_2'h_3 - g_3h_2' - g_3'h_2\,;$$

but for the same purpose as Δ and Δ' it will be convenient to retain Δ_{12}.

69. Passing now to the consideration of the aggregate of modified Δ-equations in θ_8 as the variable of reference, we notice that eleven out of thirteen have their right-hand sides solutions of $\Delta = 0$, and that therefore, when any pair is combined, they lead to a solution; for instance, from

$$\nabla_8\varepsilon = -\theta_5, \quad \nabla_8 t = \phi_2$$

we have as a solution of $\nabla_8 = 0$, i. e. of $\Delta = 0$, the quantity $\varepsilon\phi_2 + t\theta_5$. The number of such combinations is 55 $\left(= \frac{1}{2}.11.10\right)$; and the corresponding 55 solutions can be expressed in terms of the fundamental set. Some of them are merely combinations of solutions already obtained, though not yet expressed in terms of the fundamental set; some of them are solutions new in form. The relations are given in the following table, which is to be read

$$\begin{aligned} \varepsilon\theta_7 + (-q)(-\theta_5) &= \phi_2, & \varepsilon\phi + (-\gamma)(-\theta_5) &= \psi_5', \\ q\phi + (-q)\,\theta_7 &= -\lambda_7', & q\psi_5 + \alpha\theta_7 &= v_2, \end{aligned}$$

and so on:

$\circ$	$(\theta_7, -q)$	$(\phi, -\gamma)$	(ψ_5, α)	$(\psi_7, -x)$	$(\phi_2, -t)$	$(\chi_2, -l)$	$(\lambda_7', -u)$	$\lambda_7, -x$	$(\omega_2', -\tau)$	$(\upsilon_2, -\lambda)$
$(\varepsilon, -\theta_5)$	ϕ_2	ψ_5'	$-\psi$	χ_2	χ_7	ϕ_7	υ_2'	ω_2	ρ_7'	ρ_7
	(q, θ_7)	$-\lambda_7'$	υ_2	$\mathfrak{g}$	$\Delta\theta_5$	$\mathfrak{i}_1+\Delta_{12}\theta_5$	$\mathfrak{k}_{12}-\Delta_{12}\phi$	$-\Delta\phi$	$\mathfrak{i}_2+\Delta_{12}\psi_5$	$\Delta\psi_5$
		(γ, ϕ)	θ_5'	λ_7'	ω_2	υ_2	θ_7'	ψ_7'	ϕ_2'	χ_2'
			$(\alpha, -\psi_5)$	χ_2	ρ_7	ρ_7'	ϕ_2'	χ_2'	χ_7'	ϕ_7'
				(x, ψ_7)	$-\mathfrak{i}_1+\Delta_{12}\theta_5$	$\Delta'\theta_5$	$-\Delta'\phi$	$-\mathfrak{k}_{12}-\Delta_{12}\phi$	$\Delta'\psi_5$	$-\mathfrak{i}_2'+\Delta_{12}\psi_5$
					(t, ϕ_2)	$\mathfrak{k}$	$\mathfrak{i}_2'-\Delta_{12}\psi_5'$	$-\Delta\psi_5'$	$\mathfrak{g}_{12}-\Delta_{12}\psi$	$-\Delta\psi$
						(l, χ_2)	$-\Delta'\psi_5'$	$-\mathfrak{i}_2'-\Delta_{12}\psi_5'$	$-\Delta'\psi$	$-\mathfrak{g}_{12}-\Delta_{12}\psi$
							(u, λ_7')	$-\mathfrak{g}'$	$\Delta'\theta_5'$	$-\mathfrak{i}_1'+\Delta_{12}\theta_5'$
								(x, λ_7)	$\mathfrak{i}_1'+\Delta_{12}\theta_5'$	$\Delta\theta_5'$
									(τ, ω_2')	$-\mathfrak{k}'$

The quantities which occur in this table and not already defined are given by the following definitions, in which

$$\begin{aligned} \mathfrak{A} &= -h_3'(h_2 - g_3) + h_3(h_2' - g_3'), \\ \mathfrak{B} &= \quad g_2'h_3 - g_2h_3', \\ \mathfrak{C} &= -g_2'(h_2 - g_3) + g_2(h_2' - g_3'); \end{aligned}$$

it will be seen that all the functions are of the nature of Jacobians:

$$\left.\begin{aligned} \mathfrak{g} &= 2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(g_1, -h_1)^2 \\ \mathfrak{g}_{12} &= 2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(g_1, -h_1)(g_1', -h_1') \\ \mathfrak{g}' &= 2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(g_1', -h_1')^2 \end{aligned}\right\}, \quad \left.\begin{aligned} \mathfrak{k} &= 2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(a_2, a_3)^2 \\ \mathfrak{k}_{12} &= 2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(a_2, a_3)(a_2', a_3') \\ \mathfrak{k}' &= 2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(a_2', a_3')^2 \end{aligned}\right\},$$

$$\left.\begin{aligned} \mathfrak{i}_1 &= 2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(g_1, -h_1)(a_2, a_3) \\ \mathfrak{i}_1' &= 2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(g_1', -h_1')(a_2', a_3') \end{aligned}\right\}, \quad \left.\begin{aligned} \mathfrak{i}_2 &= 2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(g_1, -h_1)(a_2, a_3') \\ \mathfrak{i}_2' &= 2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(g_1', -h_1')(a_2, a_3) \end{aligned}\right\}.$$

70. The simplest integral relations among the solutions already obtained are:

$$\left.\begin{aligned} \lambda_7\theta_5 &= \theta_7\psi_5' - \phi\phi_2 \\ \lambda_7\psi_5 &= \theta_7\theta_5' - \phi v_2 \\ \lambda_7\psi_5' &= \psi_7'\theta_5 + \phi\omega_2 \\ \lambda_7\theta_5' &= \psi_7'\psi_5 + \phi\chi_2' \end{aligned}\right\}; \quad \left.\begin{aligned} \lambda_7'\theta_5 &= \psi_7\psi_5' - \phi\chi_2 \\ \lambda_7'\psi_5 &= \psi_7\theta_5' - \phi\omega_2' \\ \lambda_7'\psi_5' &= \theta_7'\theta_5 + \phi v_2' \\ \lambda_7'\theta_5' &= \theta_7'\psi_5 + \phi\phi_2' \end{aligned}\right\};$$

$$\left.\begin{aligned} \rho_7\theta_5 &= \chi_7\psi_5 + \psi\phi_2 \\ \rho_7\psi_5 &= \phi_7'\theta_5 - \psi v_2 \\ \rho_7\psi_5' &= \chi_7\theta_5' + \psi\omega_2 \\ \rho_7\theta_5' &= \phi_7'\psi_5' - \psi\chi_2' \end{aligned}\right\}, \quad \left.\begin{aligned} \rho_7'\theta_5 &= \phi_7\psi_5 + \psi\chi_2 \\ \rho_7'\psi_5 &= \chi_7'\theta_5 - \psi\omega_2' \\ \rho_7'\psi_5' &= \phi_7\theta_5' + \psi v_2' \\ \rho_7'\theta_5' &= \chi_7'\psi_5' - \psi\phi_2' \end{aligned}\right\};$$

$$\left.\begin{aligned} \theta_7\chi_7 &= \phi_2^2 - \theta_5^2\Delta \\ \theta_7\phi_7' &= v_2^2 - \psi_5^2\Delta \\ \psi_7'\chi_7 &= \omega_2^2 - \psi_5'^2\Delta \\ \psi_7'\phi_7' &= \chi_2'^2 - \theta_5'^2\Delta \end{aligned}\right\}; \quad \left.\begin{aligned} \psi_7\phi_7 &= \chi_2^2 - \theta_5^2\Delta' \\ \psi_7\chi_7' &= \omega_2'^2 - \psi_5^2\Delta' \\ \theta_7'\phi_7 &= v_2'^2 - \psi_5'^2\Delta' \\ \theta_7'\chi_7' &= \phi_2'^2 - \theta_5'^2\Delta' \end{aligned}\right\};$$

$$\left.\begin{aligned} \theta_7\psi_7' &= \lambda_7^2 - \phi^2\Delta \\ \theta_7'\psi_7 &= \lambda_7'^2 - \phi^2\Delta' \end{aligned}\right\}; \quad \left.\begin{aligned} \chi_7\phi_7' &= \rho_7^2 - \psi^2\Delta \\ \chi_7'\phi_7 &= \rho_7'^2 - \psi^2\Delta' \end{aligned}\right\};$$

$$\left.\begin{aligned} \mathfrak{g}\psi_5' - \mathfrak{g}_{12}\theta_5 &= \phi\mathfrak{i}_1 \\ \mathfrak{g}\theta_5' - \mathfrak{g}_{12}\psi_5 &= \phi\mathfrak{i}_2 \\ \mathfrak{g}_{12}\psi_5' - \mathfrak{g}'\theta_5 &= \phi\mathfrak{i}_2' \\ \mathfrak{g}_{12}\theta_5' - \mathfrak{g}'\psi_5 &= \phi\mathfrak{i}_1' \end{aligned}\right\}; \quad \left.\begin{aligned} \mathfrak{k}'\theta_5 - \mathfrak{k}_{12}\psi_5 &= \psi\mathfrak{i}_2 \\ \mathfrak{k}'\psi_5 - \mathfrak{k}_{12}\theta_5 &= \psi\mathfrak{i}_1' \\ \mathfrak{k}_{12}\psi_5' - \mathfrak{k}\theta_5' &= \psi\mathfrak{i}_2' \\ \mathfrak{k}_{12}\theta_5 - \mathfrak{k}\psi_5 &= \psi\mathfrak{i}_1 \end{aligned}\right\};$$

$$\left.\begin{aligned} \Delta'\theta_7^2 - 2\Delta_{12}\theta_7\psi_7 + \Delta\psi_7^2 &= \mathfrak{g}^2 \\ \Delta'\psi_7'^2 - 2\Delta_{12}\theta_7'\psi_7' + \Delta\theta_7'^2 &= \mathfrak{g}'^2 \\ \Delta'\chi_7^2 - 2\Delta_{12}\chi_7\phi_7 + \Delta\phi_7^2 &= \mathfrak{k}^2 \\ \Delta'\phi_7'^2 - 2\Delta_{12}\phi_7'\chi_7' + \Delta\chi_7'^2 &= \mathfrak{k}'^2 \end{aligned}\right\};$$

$$\left.\begin{aligned}
\Delta'\lambda_7^2 - 2\Delta_{12}\lambda_7\lambda_7' + \Delta\lambda_7'^2 &= \mathfrak{g}_{12}^2 + (\Delta\Delta' - \Delta_{12}^2)\,\phi^2 \\
\Delta'\rho_7^2 - 2\Delta_{12}\rho_7\rho_7' + \Delta\rho_7'^2 &= \mathfrak{h}_{12}^2 + (\Delta\Delta' - \Delta_{12}^2)\,\psi^2 \\
\Delta'\phi_2^2 - 2\Delta_{12}\phi_2\chi_2 + \Delta\chi_2^2 &= \mathfrak{i}_1^2 + (\Delta\Delta' - \Delta_{12}^2)\,\theta_5^2 \\
\Delta'\upsilon_2^2 - 2\Delta_{12}\upsilon_2\omega_2' + \Delta\omega_2'^2 &= \mathfrak{i}_2^2 + (\Delta\Delta' - \Delta_{12}^2)\,\psi_5^2 \\
\Delta'\omega_2^2 - 2\Delta_{12}\upsilon_2'\omega_2 + \Delta\upsilon_2'^2 &= \mathfrak{i}_2'^2 + (\Delta\Delta' - \Delta_{12}^2)\,\psi_5'^2 \\
\Delta'\chi_2'^2 - 2\Delta_{12}\chi_2'\phi_2' + \Delta\phi_2'^2 &= \mathfrak{i}_1'^2 + (\Delta\Delta' - \Delta_{12}^2)\,\theta_5'^2
\end{aligned}\right\};$$

The remainder are given in the following table, which is such that some equivalent value has been found for every product-pair of the twenty quantics in §66 which are of the third degree in the coefficients. There is no equivalent value for the product $\theta_7\lambda_7$; and the table is to be read, for instance, in the line λ_7,

there is no equivalent value for $\lambda_7\psi_7'$;

$$\lambda_7\chi_7 = \phi_2\omega_2 - \theta_5\psi_5'\Delta,$$

and for $\phi_2\omega_2$ a reference is made to the entry for $\lambda_7\chi_7$; and it will be noticed that the results are all of such a form that the difference of two of these product-pairs is expressible in terms of quantities each of which has at least one factor of the second degree.

71. Each of these relations between the solutions of the differential equations implies a syzygy between the corresponding concomitants of the system of two quantics, the concomitant u_x—whose leading coefficient is unity—being used, where necessary, to make the order and the class uniform in the syzygy.

72. Each of the solutions obtained determines a concomitant; the order and the class of each such function so determined are given in the following table:

ORDER IN x.	CLASS IN u.	LEADING COEFFICIENT.
1	1	$\theta_1,\ \theta_1',\ \theta_8,\ \theta_8',\ \vartheta_2,\ \vartheta_2',\ \vartheta_{12}$.
2	2	$\theta_5,\ \theta_5',\ \psi_5,\ \psi_5'$.
3	3	$\phi_2,\ \phi_2',\ \upsilon_2,\ \upsilon_2',\ \omega_2,\ \omega_2',\ \chi_2',\ \chi_2,\ \mathfrak{i}_1,\ \mathfrak{i}_1',\ \mathfrak{i}_2,\ \mathfrak{i}_2'$.
0	3	$\phi,\ \theta_7,\ \theta_7'$.
1	4	$\lambda_7,\ \lambda_7',\ \psi_7,\ \psi_7'$.
2	5	$\mathfrak{g},\ \mathfrak{g}_{12},\ \mathfrak{g}'$.
3	0	$\psi,\ \chi_7,\ \chi_7'$.
4	1	$\rho_7,\ \rho_7',\ \phi_7,\ \phi_7'$.
5	2	$\mathfrak{h},\ \mathfrak{h}_{12},\ \mathfrak{h}'$.

73. The symbolical values of the more important concomitants are as follow, the original quantics being

$$U = b_x u_\beta = c_x u_\gamma = \ldots\ldots, \quad U' = b'_x u_{\beta'} = c'_x u_{\gamma'} = \ldots\ldots,$$

and capital letters Θ_r, Φ_s, ... denoting the comcomitants which have the corresponding small letters θ_r, ϕ_s, ... with the same suffixes for leading coefficients:

$$\begin{cases} \Theta_1 = \quad b_x u_\beta; \\ \Theta'_1 = \quad b'_x u_{\beta'}; \end{cases}$$

$$\begin{cases} \Theta_8 = -\Theta_1 + u_x b_\beta; \\ \Theta'_8 = -\Theta'_1 + u_x b'_{\beta'}; \end{cases}$$

$$\begin{cases} \Theta_5 = -\Theta_1^2 + u_x c_\beta b_x u_\gamma; \\ \Psi_5 = -\Theta_1 \Theta'_1 + u_x b_{\beta'} b'_x u_\beta; \\ \Psi'_5 = -\Theta_1 \Theta'_1 + u_x b'_\beta b_x u_{\beta'}; \\ \Theta'_5 = -\Theta_1'^2 + u_x c'_{\beta'} b'_x u_{\gamma'}; \end{cases}$$

$$\begin{cases} \Theta_2 = -(bcu)(\beta\gamma x); \\ \Theta_{12} = -(bb'u)(\beta\beta' x); \\ \Theta'_2 = -(b'c'u)(\beta'\gamma' x); \end{cases}$$

$$\begin{cases} \Phi = -u_\beta u_{\beta'} (bb'u); \\ \Psi = -b_x b'_x (\beta\beta' x); \end{cases}$$

$$\begin{cases} \frac{1}{2}\Theta_7 = -d_\beta u_\gamma u_\delta (bcu); \\ \frac{1}{2}\Psi_7 = \quad u_x c_{\beta'} u_\beta u_\gamma (bb'u) - c_x u_\beta u_\gamma u_{\beta'} (bb'u); \\ \frac{1}{2}\Psi'_7 = -u_x c'_\beta u_{\beta'} u_{\gamma'} (bb'u) + c'_x u_{\beta'} u_{\gamma'} u_\beta (bb'u); \\ \frac{1}{2}\Theta'_7 = -d'_{\beta'} u_{\gamma'} u_{\delta'} (b'c'u); \end{cases}$$

$$\begin{cases} \frac{1}{2}\Lambda_7 = \quad \frac{1}{2}\Theta_8 \Phi - d_{\beta'} u_{\delta'} u_\gamma (bcu) u_x; \\ \frac{1}{2}\Lambda'_7 = -\frac{1}{2}\Theta'_8 \Phi - d'_\beta u_\delta u_{\gamma'} (b'c'u) u_x; \end{cases}$$

$$\begin{cases} \frac{1}{2}X_7 = \quad b_\gamma c_x d_x (\beta\delta x); \\ \frac{1}{2}\Phi_7 = \quad u_x b'_\gamma c_x d_x (\beta'\delta x) - b'_x c_x d_x u_\gamma (\beta'\delta x); \\ \frac{1}{2}\Phi'_7 = -u_x b_{\gamma'} c'_x d'_x (\beta\delta' x) + b_x c'_x d'_x u_{\gamma'} (\beta\delta' x); \\ \frac{1}{2}X'_7 = \quad b'_{\gamma'} c'_x d'_x (\beta'\delta' x); \end{cases}$$

$$\left\{\begin{aligned} \tfrac{1}{2}\mathrm{P}_7 &= -\tfrac{1}{2}\Theta_8\Psi + b_{\beta'}b'_x c_x(\beta\gamma x)u_x; \\ \tfrac{1}{2}\mathrm{P}'_7 &= \tfrac{1}{2}\Theta'_8\Psi + b'_\beta b_x c'_x(\beta'\gamma' x)u_x; \end{aligned}\right.$$

$$\left\{\begin{aligned} \tfrac{1}{2}\Phi_2 &= -\tfrac{1}{2}\Theta_8\Theta_5 + d_x u_\gamma(bcu)(\beta\delta x)u_x; \\ \tfrac{1}{2}\Phi'_2 &= -\tfrac{1}{2}\Theta'_8\Theta'_5 + d'_x u_{\gamma'}(b'c'u)(\beta'\delta' x)u_x; \end{aligned}\right.$$

$$\left\{\begin{aligned} \tfrac{1}{2}\Upsilon_2 &= -\tfrac{1}{2}\Theta_8\Psi_5 + b'_x u_\gamma(bcx)(\beta\beta' u)u_x; \\ \tfrac{1}{2}\Upsilon'_2 &= -\tfrac{1}{2}\Theta'_8\Psi'_5 - b_x u_{\gamma'}(b'c'x)(\beta\beta' u)u_x; \end{aligned}\right.$$

$$\left\{\begin{aligned} \tfrac{1}{2}\Omega_2 &= -\tfrac{1}{2}\Theta_8\Psi'_5 + c_x u_{\beta'}(bb'u)(\beta\gamma x)u_x; \\ \tfrac{1}{2}\Omega'_2 &= -\tfrac{1}{2}\Theta'_8\Psi_5 - c'_x u_\beta(bb'u)(\beta'\gamma' x)u_x; \end{aligned}\right.$$

$$\left\{\begin{aligned} \tfrac{1}{2}\mathrm{X}_2 &= -\tfrac{1}{2}\Theta'_8\Theta_5 - c_x u_\beta(bb'u)(\beta'\gamma x)u_x; \\ \tfrac{1}{2}\mathrm{X}'_2 &= -\tfrac{1}{2}\Theta_8\Theta'_5 + c'_x u_{\beta'}(bb'u)(\beta\gamma' x)u_x.\text{*} \end{aligned}\right.$$

III.—*The Quadro-Linear Quantic.*

74. This may be taken in the form

$$\begin{aligned} &(ax_1^2 + bx_2^2 + cx_3^2 + 2fx_2x_3 + 2gx_3x_1 + 2hx_1x_2)u_1 \\ &+ (a'x_1^2 + b'x_2^2 + c'x_3^2 + 2f'x_2x_3 + 2g'x_3x_1 + 2h'x_1x_2)u_2 \\ &+ (a''x_1^2 + b''x_2^2 + c''x_3^2 + 2f''x_2x_3 + 2g''x_3x_1 + 2h''x_1x_2)u_3, \end{aligned}$$

which is symbolically represented by $a_x^2u_a$. The characteristic equations are

$$\begin{aligned} D_1 = 2f\frac{\partial}{\partial b} + c\frac{\partial}{\partial f} + g\frac{\partial}{\partial h} + 2f'\frac{\partial}{\partial b'} + c'\frac{\partial}{\partial f'} + g'\frac{\partial}{\partial h'} - a'\frac{\partial}{\partial a''} - g'\frac{\partial}{\partial g''} \\ - c'\frac{\partial}{\partial c''} + (2f'' - b')\frac{\partial}{\partial b''} + (c'' - f')\frac{\partial}{\partial f''} + (g'' - h')\frac{\partial}{\partial h''} = 0, \end{aligned}$$

$$\begin{aligned} \Delta = D_6 = 2f\frac{\partial}{\partial c} + b\frac{\partial}{\partial f} + h\frac{\partial}{\partial g} + 2f''\frac{\partial}{\partial c''} + b''\frac{\partial}{\partial f''} + h''\frac{\partial}{\partial g''} - a''\frac{\partial}{\partial a'} \\ - b''\frac{\partial}{\partial b'} - h''\frac{\partial}{\partial h'} + (2f' - c'')\frac{\partial}{\partial c'} + (b' - f'')\frac{\partial}{\partial f'} + (h' - g'')\frac{\partial}{\partial g'} = 0. \end{aligned}$$

---

*I have not worked out in any detail the forms for *three lineo-linear* quantics; but it is interesting to see that the cubic determinant formed of the coefficients

$$\begin{matrix} a_1, h_1, g_1; & a'_1, h'_1, g'_1; & a''_1, h''_1, g''_1 \\ a_2, h_2, g_2; & a'_2, h'_2, g'_2; & a''_2, h''_2, g''_2 \\ a_3, h_3, g_3; & a'_3, h'_3, g'_3; & a''_3, h''_3, g''_3 \end{matrix}$$

as the three "strata" (see Scott's "Determinants," Chap. VII), is a leading coefficient of a concomitant.

All the *simultaneous solutions can be expressed in terms of fifteen independent simultaneous solutions.*

75. It appears that, of the seventeen equations subsidiary to D_1, six integrals are at once given by a, g, c, a', g', c'; as in previous cases, we have a choice of variables of reference in g or a'.

The systems of integrals and the modified Δ-equations are formed as in the preceding cases, with the following results:

Quantities, being solutions of $D_1 = 0$ but not of $\Delta = 0$ and occurring in the equations, are

$$\left.\begin{array}{lll}\theta_2 = a', & \theta_6 = g', & \theta_8 = c' \\ \theta_{11} = f' + c'', & \theta_5 = g, & \theta_3 = c\end{array}\right\},$$

$$\left.\begin{array}{l}\theta_4 = (f,\ c\between a',\ a'') \\ \theta_7 = (\lambda,\ g'\between a',\ a'') \\ \theta_9 = (\mu,\ c'\between a',\ a'')\end{array}\right\},\quad \left.\begin{array}{l}\phi_4 = (f,\ c\between g,\ -h) \\ \phi_7 = (\lambda,\ g'\between g,\ -h) \\ \phi_9 = (\mu,\ c'\between g,\ -h)\end{array}\right\},$$

$$\left.\begin{array}{l}\theta_{10} = (\rho,\ \mu,\ c'\between a',\ a'')^2 \\ \psi_{10} = (\rho,\ \mu,\ c'\between a',\ a''\between g,\ -h) \\ \phi_{10} = (\rho,\ \mu,\ c'\between g,\ -h)^2\end{array}\right\},$$

where the symbols λ, μ, ρ are defined by the equations

$$\lambda = \tfrac{1}{2}(h' - g''),\quad \mu = \tfrac{1}{8}(2f' - c''),\quad \rho = \tfrac{1}{3}(b' - 2f'').$$

Quantities, being solutions of both $D_1 = 0$ and $\Delta = 0$ and occurring in the equations, are, in addition to $y_1 = a$, given by

$$\left.\begin{array}{l}y_3 = (b,\ f,\ c\between a',\ a'')^2 \\ \xi_3 = (b,\ f,\ c\between a',\ a''\between g,\ -h) \\ z_3 = (b,\ f,\ c\between g,\ -h)^2\end{array}\right\};\quad \left.\begin{array}{l}y_4 = (-h'',\ \lambda,\ g'\between a',\ a'')^2 \\ \xi_4 = (-h'',\ \lambda,\ g'\between a',\ a''\between g,\ -h) \\ z_4 = (-h'',\ \lambda,\ g'\between g,\ -h)^2\end{array}\right\};$$

$$\left.\begin{array}{l}y_5 = (-b'',\ \rho,\ \mu,\ c'\between a',\ a'')^3 \\ \eta_5 = (-b'',\ \rho,\ \mu,\ c'\between a',\ a'')^2(g,\ -h) \\ \xi_5 = (-b'',\ \rho,\ \mu,\ c'\between a',\ a''\between g,\ -h)^2 \\ z_5 = (-b'',\ \rho,\ \mu,\ c'\between g,\ -h)^3\end{array}\right\};\quad \left.\begin{array}{l}y_8 = ha' + ga'' \\ y_7 = h' + g''\end{array}\right\};\quad \left.\begin{array}{l}y_6 = (b' + f'',\ f' + c''\between a',\ a'') \\ z_6 = (b' + f'',\ f' + c''\between g,\ -h)\end{array}\right\}.$$

And it is not difficult to see that, when θ_2 is the variable of reference, the

seventeen integrals of the equations subsidiary to D_1, being

$$y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ y_7;\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6,\ \theta_7,\ \theta_8,\ \theta_9,\ \theta_{10},\ \theta_{11},$$

are independent of one another.

76. The modified Δ-equations are

$\theta_2^2\Delta = \nabla$	$\theta_5^2\Delta = \nabla'$
$\nabla i = 2k$	$\nabla' i' = 2k'$
$\nabla k = y_2$	$\nabla' k' = z_2$
$\nabla l = y_3$	$\nabla' l' = -y_3$
$\nabla p = 2q$	$\nabla' p' = 2q'$
$\nabla q = y_4$	$\nabla' q' = z_4$
$\nabla r = 3s$	$\nabla' r' = 3s'$
$\nabla s = 2t$	$\nabla' s' = 2t'$
$\nabla t = y_5$	$\nabla' t' = z_5$
$\nabla v = y_6$	$\nabla' v' = z_6$
$\nabla \varkappa = \xi_3$	$\nabla' \varkappa' = \xi_3$
$\nabla \chi = \xi_4$	$\nabla' \chi' = \xi_4$
$\nabla \sigma = 2\omega$	$\nabla' \sigma' = 2\omega'$
$\nabla \omega = \eta_5$	$\nabla' \omega' = \xi_5$
$\nabla \tau = \xi_5$	$\nabla' \tau' = \eta_5$

where

$$\left.\begin{aligned}
\theta_5 &= l\theta_2, \qquad \theta_2 = l'\theta_5; \\
\theta_3 &= i\theta_2^2 = i'\theta_5^2 \\
\theta_4 &= k\theta_2 = \varkappa'\theta_5 \\
\phi_4 &= \varkappa\theta_2 = k'\theta_5 \\
\theta_6 &= p\theta_2^2 = p'\theta_5^2 \\
\theta_7 &= q\theta_2 = \chi'\theta_5 \\
\phi_7 &= \chi\theta_2 = q'\theta_5 \\
\theta_8 &= r\theta_2^3 = r'\theta_5^3 \\
\theta_9 &= s\theta_2^2 = \sigma'\theta_5^2 \\
\theta_{10} &= t\theta_2 = \tau'\theta_5 \\
\phi_9 &= \sigma\theta_2^2 = s'\theta_5^2 \\
\phi_{10} &= \tau\theta_2 = t'\theta_5 \\
\theta_{11} &= v\theta_2 = v'\theta_5 \\
\psi_{10} &= \omega\theta_2 = \omega'\theta_5
\end{aligned}\right\},$$

and the first nine equations are the independent equations for each of the variables of reference.

77. Taking first the set of nine independent equations in θ_2 as the variables of reference, eight independent integrals (the necessary number) are given by

$$
\begin{aligned}
\delta_2 &= k^2 - iy_2 && = f^2 - bc, \\
\delta_4 &= q^2 - py_4 && = \lambda^2 + g'h'', \\
h_5 &= t^2 - sy_5 && = (\rho^2 + \mu b'',\ \rho\mu + b''c',\ \rho c' - \mu^2 \between a', a'')^2, \\
\phi_5 &= ry_5^2 - 3sty_5 + 2t^3 && = \boxed{\begin{array}{l|l|l|l} b''^2c' & -b''\rho c' & b''\mu c' & b''c'^2 \\ +3b''\rho\mu & +2b''\mu^2 & +2\rho^2c' & +3\rho\mu c' \\ +2\rho^3 & +\rho^2\mu & -\rho\mu^2 & -2\mu^3 \end{array}} \between a', a'')^3, \\
\varepsilon_2 &= ly_2 - ky_3 && = (gb - fh,\ gf - ch \between a', a''), \\
\varepsilon_4 &= ly_4 - qy_3 && = (-gh'' - \lambda h,\ g\lambda - hg' \between a', a''), \\
\eta_5 &= ly_5 - ty_3 && = (-b'',\ \rho,\ \mu,\ c' \between a', a'')^2(g, -h), \\
z_6 &= ly_6 - vy_3 && = (b' + f'',\ f' + c'' \between g, -h).
\end{aligned}
$$

Here δ_2 and δ_4 are the discriminants of y_2 and y_4, regarded as binary quadratics in a' and a'' as variables; h_5 is the Hessian (with changed sign) of y_5 similarly regarded as a binary cubic, and ϕ_5 is its cubicovariant; ξ_2, ξ_4, η_5, z_6 are the Jacobians of y_3 with y_2, y_4, y_5, y_6 respectively, similarly regarded as binary quantics. And if we take

$$\delta_5 = b''^2c'^2 + 6b''\rho\mu c' - 4b''\mu^3 + 4\rho^3c' - 3\rho^2\mu^2,$$

being the discriminant of y_5, regarded as a binary cubic, we have

$$\phi_5^2 = \delta_5 y_5^2 + 4h_5^3,$$

so that δ_5 is a simultaneous solution and it may replace ϕ_5.

We thus have the result:

Every simultaneous solution of the two characteristic equations can be expressed in terms of the fifteen independent simultaneous solutions

$$y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ y_7,\ \delta_2,\ \delta_4,\ h_5,\ \delta_5,\ \xi_2,\ \xi_4,\ \eta_5,\ z_6;$$

and every concomitant of the quadro-linear quantic can be expressed in terms of the fifteen concomitants which have respectively these fifteen quantities for their leading coefficients.

78. Taking next the set of nine independent equations in θ_5 as the variable of reference, the corresponding eight integrals are

$$
\begin{aligned}
\delta_2 &= k'^2 - i'z_2 = f^2 - bc,\\
\delta_4 &= q'^2 - p'z_4 = \lambda^2 + g'h'',\\
h_5' &= t'^2 - s'z_5 = (\rho^2 + \mu b'',\ \rho\mu + b''c',\ \rho c' - \mu^2 \between g, -h)^2,\\
[\phi_5' &= r'z_5^2 - 3s't'z_5 + 2t'^3, \text{ replaced as before by}]\\
\delta_5 &= \qquad\qquad = b''^2c'^2 + 6b''\rho\mu c' - 4b''\mu^3 + 4\rho^3c' - 3\rho^2\mu^2,\\
\xi_4 &= l'z_4 + q'y_3 = (-h''a' + \lambda a'',\ \lambda a' + ha'' \between g, -h),\\
\xi_2 &= l'z_2 + k'y_3 = (ba' + fa'',\ fa' + ca'' \between g, -h),\\
\xi_5 &= l'z_5 + t'y_3 = (-b'',\ \rho,\ \mu,\ c' \between g, -h)^2(a', a''),\\
y_6 &= l'z_6 + v'y_3 = (b' + f'',\ f' + c'' \between a', a'');
\end{aligned}
$$

these quantities bearing similar relations to the quantities z, viewed as binary quantics in g and $-h$ as variables. Hence we see:

Every simultaneous solution of the two characteristic equations can also be expressed in terms of the fifteen independent simultaneous solutions

$$y_1,\ z_2,\ y_3,\ z_4,\ z_5,\ z_6,\ y_7,\ \delta_2,\ \delta_4,\ h_5',\ \delta_5,\ \xi_2;\ \xi_4,\ \xi_5;\ y_6;$$

and every concomitant of the quadro-linear quantic can also be expressed in terms of the fifteen concomitants which have respectively these fifteen quantities for their leading coefficients.

79 To obtain the order and the class of each of the concomitants, we may use either the method of developing operators in §7 and 8; or we may obtain the symbolical forms, the umbral coefficient-combinations being given in the accompanying table. For any one of them such as z_5, we first change the quantities b'', ρ, μ, c' (its coefficients regarded as a binary form) into umbral combinations, so that

	α_1^2	α_2^2	α_3^2	$\alpha_2\alpha_3$	$\alpha_3\alpha_1$	$\alpha_1\alpha_2$
a_1	a	b	c	f	g	h
a_2	a'	b'	c'	f'	g'	h'
a_3	a''	b''	c''	f''	g''	h''

$$
\begin{aligned}
z_5 &= -g^3a_2^2a_3 - g^2h(a_3^2a_2 - 2a_2a_3\alpha_3) + gh^2(2a_2a_3\alpha_2 - a_3^2\alpha_3) - h^3a_3^2\alpha_2\\
&= -(a_2g - a_3h)^2(\alpha_3g + \alpha_2h).
\end{aligned}
$$

Now we have also

$$
\begin{aligned}
a_2g - a_3h &= b_1\beta_1(a_2b_3 - a_3b_2) = b_1\beta_1(a_2b_3),\\
\alpha_3g + \alpha_2h &= d_1\delta_1(a_3d_3 + a_2d_2) = d_1\delta_1(d_a - d_1\alpha_1),
\end{aligned}
$$

so that

$$z_5 = -b_1\beta_1(a_2b_3)\,c_1\gamma_1(a_2c_3)\,d_1\delta_1(d_a - d_1\alpha_1)$$

and therefore

$$Z_5 = -b_xc_xd_xu_\beta u_\gamma u_\delta(abu)(acu)\,d_au_\alpha + b_xc_xu_au_\beta u_\gamma d_x^2u_\delta(abu)(acu).$$

The second term is seen to be resoluble, for $d_x^2 u_\delta$ is a factor; it is in fact equal to what is called $Y_1 Z_2$: and thus Z_5 effectively determines a concomitant $d_\alpha b_x c_x d_x u_\beta u_\gamma u_\delta (abu)(acu)$.

The symbolical expressions are here given for all except h_5, h_5' and δ_5, which are long and complicated in their symbolical form. The order in x and the class in u are 8 and 4, 4 and 6, 4 and 6 respectively for these three; and for the others are immediately evident from an inspection of their values:

$$
\begin{aligned}
Y_1 &= y_1 x_1^2 u_1 + \ldots = a_x^2 u_\alpha, \\
Y_3 &= y_3 x_1^4 u_1^2 + \ldots = -Y_1^2 + a_\beta a_x b_x^2 u_\alpha u_x, \\
Y_7 &= y_7 x_1^2 u_1 + \ldots = -Y_1 + a_x u_\alpha a_\alpha, \\
Y_2 &= y_2 x_1^6 u_1^3 + \ldots = -Y_1^3 - 2 Y_1 Y_3 + a_\beta a_\gamma b_x^2 c_x^2 u_\alpha u_x^2, \\
Y_4 &= y_4 x_1^6 \quad + \ldots = a_\beta a_x b_x^2 c_x^2 (\alpha\gamma x), \\
Y_5 &= y_5 x_1^8 u_1 + \ldots = -2 Y_1 Y_4 + a_\beta a_\gamma b_x^2 c_x^2 d_x^2 (\alpha\delta x) u_x, \\
Y_6 &= y_6 x_1^4 u_1^2 + \ldots = -Y_3 - Y_1 Y_7 - Y_1^2 + a_\alpha a_\beta b_x^2 u_x^2, \\
Z_2 &= z_2 x_1^2 u_1^5 + \ldots = b_x c_x u_\alpha u_\beta u_\gamma (abu)(acu), \\
Z_4 &= z_4 x_1^3 u_1^3 + \ldots = c_\beta a_x b_x c_x u_\alpha u_\gamma (abu), \\
Z_5 &= z_5 x_1^4 u_1^6 + \ldots = Y_1 Z_2 - d_\alpha b_x c_x d_x u_\beta u_\gamma u_\delta (abu)(acu) u_x, \\
Z_6 &= z_6 x_1 u_1^3 + \ldots = a_\alpha b_x u_\beta (abu), \\
\Delta_2 &= \delta_2 u_1^4 \quad + \ldots = -\frac{1}{2} u_\alpha u_\beta (abu)^2, \\
\Delta_4 &= \delta_4 x_1^4 u_1^2 + \ldots = \frac{1}{4} Y_7^2 - \frac{1}{2} a_x b_x (abu)(\alpha\beta x) u_x, \\
\Xi_2 &= \xi_2 x_1^3 u_1^3 + \ldots = a_\gamma b_x c_x^2 u_\alpha u_\beta (abu), \\
\Xi_4 &= \xi_4 x_1^6 u_1^3 + \ldots = -\frac{1}{2} Y_7 Y_3 + a_x b_x^2 c_x u_\gamma (acu)(\alpha\beta x) u_x, \\
\Xi_5 &= \xi_5 x_1^5 u_1^4 + \ldots = \frac{2}{3} Y_3 Z_6 + b_x c_x d_x^2 u_\beta u_\gamma (abu)(acu)(\alpha\delta x), \\
\mathrm{H}_5 &= \eta_5 x_1^8 u_1^4 + \ldots = \frac{2}{3} Y_3 Y_6 + 3 Y_1 Y_2 + 3 Y_1^2 Y_3 - Y_1^4 - a_\beta a_\gamma d_\alpha b_x^2 c_x^2 d_x u_\delta u_x^3.
\end{aligned}
$$

80. The operators which serve for the full development of the concomitants in powers of the variables from the leading coefficients are:

$$
\left.
\begin{aligned}
D_3 = 2h \frac{\partial}{\partial a} + b \frac{\partial}{\partial h} + f \frac{\partial}{\partial g} + 2h'' \frac{\partial}{\partial a''} + b'' \frac{\partial}{\partial h''} + f'' \frac{\partial}{\partial g''} - b \frac{\partial}{\partial b'} \\
- f \frac{\partial}{\partial f'} - c \frac{\partial}{\partial c'} + (2h' - a) \frac{\partial}{\partial a'} + (b' - h) \frac{\partial}{\partial h'} + (f' - g) \frac{\partial}{\partial g'} \\
D_5 = 2g \frac{\partial}{\partial a} + f \frac{\partial}{\partial h} + c \frac{\partial}{\partial g} + 2g' \frac{\partial}{\partial a'} + f' \frac{\partial}{\partial h'} + c' \frac{\partial}{\partial g'} - b \frac{\partial}{\partial b''} \\
- f \frac{\partial}{\partial f''} - c \frac{\partial}{\partial c''} + (2g'' - a) \frac{\partial}{\partial a''} + (f'' - h) \frac{\partial}{\partial h''} + (c'' - g) \frac{\partial}{\partial g''}
\end{aligned}
\right\},
$$

so far as powers of the x-variables are concerned; and

$$\left.\begin{array}{l} D_3 = 2g'\dfrac{\partial}{\partial c'} + h'\dfrac{\partial}{\partial f'} + a'\dfrac{\partial}{\partial g'} + 2g''\dfrac{\partial}{\partial c''} + h''\dfrac{\partial}{\partial f''} + a''\dfrac{\partial}{\partial g''} - a''\dfrac{\partial}{\partial a} \\ \qquad - h''\dfrac{\partial}{\partial h} - b''\dfrac{\partial}{\partial b} + (2g - c'')\dfrac{\partial}{\partial c} + (h - f'')\dfrac{\partial}{\partial f} + (a - g'')\dfrac{\partial}{\partial g} \\ D_4 = 2h'\dfrac{\partial}{\partial b'} + a'\dfrac{\partial}{\partial h'} + g'\dfrac{\partial}{\partial f'} + 2h''\dfrac{\partial}{\partial b''} + a''\dfrac{\partial}{\partial h''} + g''\dfrac{\partial}{\partial f''} - a'\dfrac{\partial}{\partial a} \\ \qquad - g'\dfrac{\partial}{\partial g} - c'\dfrac{\partial}{\partial c} + (2h - b')\dfrac{\partial}{\partial b} + (a - h')\dfrac{\partial}{\partial h} + (g - f')\dfrac{\partial}{\partial f} \end{array}\right\},$$

so far as powers of the u-variables are concerned.

82. Other solutions of the modified Δ-equations can be obtained, different in form but of course not algebraically independent; of these the most important are the set of three

$$\begin{aligned} \delta_{24} &= 2kq - iy_4 - py_2 = 2f\lambda + ch'' - bg', \\ \delta_{25} &= 2kt - sy_2 - iy_5 = (2f\rho - \mu b + b''c)\,a' + (2f\mu - bc' - \rho c)\,a'', \\ \delta_{45} &= 2qt - py_5 - sy_4 = (2\lambda\rho + b''g' + \mu h'')\,a' + (2\lambda\mu - g'\rho + c'h'')\,a'', \end{aligned}$$

which are respectively intermediaries between δ_2 and δ_4, δ_2 and δ_5, δ_4 and δ_5; and the set of functions of Jacobian form similar to those in §69.

83. The last statement is justified by the theorem:

The Jacobian of any two simultaneous solutions regarded as binary forms in a' and a'' is also a solution; and similarly for the Jacobians of solutions regarded as binary forms in g and $-h$.

The proof for the two cases is very much the same; taking it for the former, let U and V be two solutions of orders m and n respectively in a' and a'', and let J be their Jacobian. Then

$$\begin{aligned} mnJa' &= a'\left(\frac{\partial U}{\partial a''}\frac{\partial V}{\partial a'} - \frac{\partial U}{\partial a'}\frac{\partial V}{\partial a''}\right) \\ &= nV\frac{\partial U}{\partial a''} - mU\frac{\partial V}{\partial a''}. \end{aligned}$$

Now on looking at the groups of subsidiary quantities and comparing them with the equations, it appears that (1) for every simultaneous solution U, the associated quantity $\dfrac{\partial U}{\partial a''}$ satisfies $D_1 = 0$; hence, as a', U, V, $\dfrac{\partial U}{\partial a''}$, $\dfrac{\partial V}{\partial a''}$ all satisfy

$D_1 = 0$, it follows that J satisfies that equation; and (2) the modified Δ-equation is

$$\nabla\left(\frac{1}{ma'}\,\frac{\partial U}{\partial a''}\right) = U,$$

so that

$$\nabla J = V.U - U.V = 0,$$

and therefore J satisfies $\Delta = 0$. This proves the proposition for the former part; the latter part is similarly proved. Examples occur in ξ_3, ξ_4, ξ_5, η_5.

84. A set of dependent concomitants will thus be obtained having each as its leading coefficient a Jacobian in a' and a'' or in g and $-h$ of any two leading coefficients already obtained.

The relations among the various solutions will be similar to those previously obtained; only five examples will here be given, being those which connect the system of §77 with that of §78 immediately succeeding. They are

$$\begin{aligned} y_2 z_2 &= \xi_2^2 - \delta_2 y_3^2, \\ y_4 z_4 &= \xi_4^2 - \delta_4 y_3^2, \\ \left.\begin{aligned} y_5 \xi_5 &= \eta_5^2 - h_5 y_3^2 \\ z_5 \eta_5 &= \xi_5^2 - h_5' y_3^2 \end{aligned}\right\}&, \\ h_5 h_5' &= g_5^2 + \tfrac{1}{4}\,\delta_5 y_3^2, \end{aligned}$$

where g_5, an intermediate between h_5 and h_5', being equal to

$$\left(\rho^2 + \mu b'',\ \tfrac{1}{2}\rho\mu + \tfrac{1}{2}b''c',\ \rho c' - \mu^2 \between a',\ a'' \between g,\ -h\right);$$

is determined by the relation

$$2y_3^2 g_5 = \eta_5 \xi_5 - y_5 z_5.$$

IV.—*The Cubo-Linear Quantic.*

85. This I take in the form

$$\begin{aligned} &u_1\,(ax_1^3 + 3hx_1^2x_2 + 3gx_1^2x_3 + 3bx_2^2x_1 + 3cx_3^2x_1 + 6fx_1x_2x_3 + ix_2^3 + 3jx_2^2x_3 + 3kx_2x_3^2 + lx_3^3) \\ +\,&u_2\,(a'x_1^3 + 3h'x_1^2x_2 + 3g'x_1^2x_3 + 3b'x_2^2x_1 + 3c'x_3^2x_1 + 6f'x_1x_2x_3 + i'x_2^3 + 3j'x_2^2x_3 + 3k'x_2x_3^2 + l'x_3^3) \\ +\,&u_3\,(a''x_1^3 + 3h''x_1^2x_2 + 3g''x_1^2x_3 + 3b''x_2^2x_1 + 3c''x_3^2x_1 + 6f''x_1x_2x_3 \\ &\qquad\qquad + i''x_2^3 + 3j''x_2^2x_3 + 3k''x_2x_3^2 + l''x_3^3) \end{aligned}$$

instead of a form such that the coefficient of u_1 is a uni-ternary cubic with coefficients as in Cayley's Third Memoir; the advantage being that all the analysis

of the quadro-linear quantic, as far as it goes, is valid here without any change. The characteristic equations are

$$\begin{aligned}D_1 = 3j\frac{\partial}{\partial i} + 2k\frac{\partial}{\partial j} + l\frac{\partial}{\partial k} + 3j'\frac{\partial}{\partial i'} + 2k'\frac{\partial}{\partial j'} + l'\frac{\partial}{\partial k'} + (3j'' - i)\frac{\partial}{\partial i''}\\ + (2k'' - j')\frac{\partial}{\partial j''} + (l'' - k')\frac{\partial}{\partial k''} + 2f\frac{\partial}{\partial b} + c\frac{\partial}{\partial f} + g\frac{\partial}{\partial h}\\ + 2f'\frac{\partial}{\partial b'} + c'\frac{\partial}{\partial f'} + g'\frac{\partial}{\partial h'} - a'\frac{\partial}{\partial a''} - g'\frac{\partial}{\partial g''} - c'\frac{\partial}{\partial c''} - l'\frac{\partial}{\partial l''}\\ + (2f'' - b')\frac{\partial}{\partial b''} + (c'' - f')\frac{\partial}{\partial f''} + (g'' - h')\frac{\partial}{\partial h''} = 0,\end{aligned}$$

and

$$\begin{aligned}\Delta = D_6 = 3k\frac{\partial}{\partial l} + 2j\frac{\partial}{\partial k} + i\frac{\partial}{\partial j} + 3k''\frac{\partial}{\partial l''} + 2j''\frac{\partial}{\partial k''} + i''\frac{\partial}{\partial j''} + (3k' - l'')\frac{\partial}{\partial l'}\\ + (2j' - k'')\frac{\partial}{\partial k'} + (i' - j'')\frac{\partial}{\partial j'} + 2f\frac{\partial}{\partial c} + b\frac{\partial}{\partial f} + h\frac{\partial}{\partial g}\\ + 2f''\frac{\partial}{\partial c''} + b''\frac{\partial}{\partial f''} + h''\frac{\partial}{\partial g''} - a''\frac{\partial}{\partial a'} - b''\frac{\partial}{\partial b'} - h''\frac{\partial}{\partial h'} - i''\frac{\partial}{\partial i'}\\ + (2f' - c'')\frac{\partial}{\partial c'} + (b' - f'')\frac{\partial}{\partial f'} + (h' - g'')\frac{\partial}{\partial g'} = 0.\end{aligned}$$

86. In addition to the quantities formed with a' as variable of reference which (§75) were for the quadro-linear quantic solutions of $D_1 = 0$ but not of $\Delta = 0$, viz. θ_r (where $r = 2, 3, \ldots, 10$) and the quantities which were for that same quantic solutions of $D_1 = 0$ and of $\Delta = 0$, viz. y_s (where $s = 1, 2, \ldots, 7$), all of which occupy similar positions in the construction of the equations for the present case—there are the additional θ-quantities, solutions of $D_1 = 0$ but not of $\Delta = 0$, given by

$$\begin{cases}\theta_{12} = l,\\ \theta_{13} = ka' + la'',\\ \theta_{14} = ja'^2 + 2ka'a'' + la''^2,\end{cases}\qquad \begin{cases}\theta_{15} = l',\\ \theta_{16} = \varepsilon a' + l'a'',\\ \theta_{17} = \pi a'^2 + 2\varepsilon a'a'' + l'a''^2,\\ \theta_{18} = \sigma a'^3 + 3\pi a'^2 a'' + 3\varepsilon a'a''^2 + l'a''^3,\end{cases}$$

$$\begin{cases}\theta_{19} = k' + l'',\\ \theta_{20} = (j' + k'')a' + (k' + l'')a'',\end{cases}$$

where $\varepsilon = \frac{1}{4}(3k' - l'')$, $\pi = \frac{1}{4}(2j' - 2k'')$, $\sigma = \frac{1}{4}(i' - 3j'')$; and there are the

additional y-quantities, solutions of $D_1 = 0$ and of $\Delta = 0$, given by

$$\begin{aligned} y_8 &= ia'^3 + 3ja'^2a'' + 3ka'a''^2 + la''^3 = (i, j, k, l \between a', a'')^3, \\ y_9 &= (-i'', \sigma, \pi, \varepsilon, l'' \between a', a'')^4, \\ y_{10} &= (i' + j'', j' + k'', k' + l'' \between a', a'')^2. \end{aligned}$$

And the modified Δ-equations additional to those in the first column $(\theta_2^3\Delta = \nabla)$ of the table in §76 are

$$\begin{aligned} \nabla(\theta_{12}\theta_2^{-3}) &= 3\theta_{13}\theta_2^{-2}, \\ \nabla(\theta_{13}\theta_2^{-2}) &= 2\theta_{14}\theta_2^{-1}, \\ \nabla(\theta_{14}\theta_2^{-1}) &= y_8, \\ \nabla(\theta_{15}\theta_2^{-4}) &= 4\theta_{16}\theta_2^{-3}, \\ \nabla(\theta_{16}\theta_2^{-3}) &= 3\theta_{17}\theta_2^{-2}, \\ \nabla(\theta_{17}\theta_2^{-2}) &= 2\theta_{18}\theta_2^{-1}, \\ \nabla(\theta_{18}\theta_2^{-1}) &= y_9, \\ \nabla(\theta_{19}\theta_2^{-2}) &= 2\theta_{20}\theta_2^{-1}, \\ \nabla(\theta_{20}\theta_2^{-1}) &= y_{10}. \end{aligned}$$

87. Without proceeding to the formation of the modified Δ-equations when $g(=\theta_5)$ is taken as the variable of reference, or to the formation of the dependent equations in all the subsidiary quantities which arise in the two cases of θ_2 and of θ_5 respectively as variable of reference, these equations are sufficient to give the algebraically independent concomitants in terms of which all others can be expressed; but the simplest set of independent solutions, though complete functionally, form a system much less complete in point of form and syzygetic irreducibility than in preceding cases. This, however, is not important from our point of view, the purpose being the formation of an algebraically complete system. Such a system is given by:

(i) The quantics in a' and a'' as variables, viz.:

two of order zero in a' and a'', being y_1, y_7;
two of order one in a' and a'', being y_3, y_6;
three of order two in a' and a'', being y_2, y_4, y_{10};
two of order three in a' and a'', being y_5, y_8;
one of order four in a' and a'', being y_9;

(ii) the algebraically independent concomitants of each of these, taken singly, viz.:

the discriminant of each of the three y_2, y_4, y_{10};

the Hessian and the cubicovariant (or the discriminant, replacing the cubicovariant) of each of the two y_5, y_8;

the Hessian, the quadrinvariant and the cubicovariant (or the cubinvariant, replacing the cubicovariant) of y_9;

(iii) the Jacobian of y_3 with each of the seven y_6, y_2, y_4, y_{10}, y_5, y_8, y_9 in turn.

88. The total number in the system is thus 27, being less by three than the number of constants in the original quantic; this is the (§18) required number.

The order and the class of each of the concomitants determined by a leading coefficient will be determined subsequently (§§106–112) for the general biternary quantic.

V.—*The Cubo-Cubic Quantic.*

89. Without taking in separate detail the cases in order of simplicity after the last, viz. the lineo-quadratic, the lineo-cubic, quadro-quadratic and quadro-cubic, I pass on to the cubo-cubic, merely giving the equations on account of the mass of algebra. From the form in which the coefficients are taken for the present quantic, the equations apply to these omitted cases so far as in such omitted quantics the coefficients occur.

The quantic is taken in the form

$$\begin{aligned} & u_1^3 U_{00} \\ & + 3u_1^2 u_2 U_{10} + 3u_1^2 u_3 U_{01} \\ & + 3u_1 u_2^2 U_{20} + 6u_1 u_2 u_3 U_{11} + 3u_1 u_3^2 U_{02} \\ & + u_2^3 U_{30} \quad + 3u_2^2 u_3 U_{21} \quad + 3u_2 u_3^2 U_{12} + u_3^3 U_{03}; \end{aligned}$$

and the coefficients U in this quantic are

$$\begin{aligned} & ax_1^3 \\ & + 3hx_1^2 x_2 + 3gx_1^2 x_3 \\ & + 3bx_1 x_2^2 + 6fx_1 x_2 x_3 + cx_1 x_3^2 \\ & + ix_2^3 \quad + 3jx_2^2 x_3 \quad + 3kx_2 x_3^2 + lx_3^3, \end{aligned}$$

the literal coefficients a, h, g, ... being supposed to have the same indicative suffix as the quantity U in which they occur.

90. The subsidiary equations of the characteristic $D_1 = 0$ are

$$
\begin{aligned}
&\frac{da_{00}}{0} = \frac{dh_{00}}{g_{00}} = \frac{dg_{00}}{0} = \frac{db_{00}}{2f_{00}} = \frac{df_{00}}{c_{00}} = \frac{dc_{00}}{0} = \frac{di_{00}}{3j_{00}} = \frac{dj_{00}}{2k_{00}} = \frac{dk_{00}}{l_{00}} = \frac{dl_{00}}{0} \\
= &\frac{da_{10}}{0} = \frac{dh_{10}}{g_{10}} = \frac{dg_{10}}{0} = \frac{db_{10}}{2f_{10}} = \frac{df_{10}}{c_{10}} = \frac{dc_{10}}{0} = \frac{di_{10}}{3j_{10}} = \frac{dj_{10}}{2k_{10}} = \frac{dk_{10}}{l_{10}} = \frac{dl_{10}}{0} \\
= &\frac{da_{01}}{-a_{10}} = \frac{dh_{01}}{g_{01}-h_{10}} = \frac{dg_{01}}{-g_{10}} = \frac{db_{01}}{2f_{01}-b_{10}} = \frac{df_{01}}{c_{01}-f_{10}} = \frac{dc_{01}}{-c_{10}} = \frac{di_{01}}{3j_{01}-i_{10}} = \frac{dj_{01}}{2k_{01}-j_{10}} = \frac{dk_{01}}{l_{01}-k_{10}} = \frac{dl_{01}}{-l_{10}} \\
= &\frac{da_{20}}{0} = \frac{dh_{20}}{g_{20}} = \frac{dg_{20}}{0} = \frac{db_{20}}{2f_{20}} = \frac{df_{20}}{c_{20}} = \frac{dc_{20}}{0} = \frac{di_{20}}{3j_{20}} = \frac{dj_{20}}{2k_{20}} = \frac{dk_{20}}{l_{20}} = \frac{dl_{20}}{0} \\
= &\frac{da_{11}}{-a_{20}} = \frac{dh_{11}}{g_{11}-h_{20}} = \frac{dg_{11}}{-g_{20}} = \frac{db_{11}}{2f_{11}-b_{20}} = \frac{df_{11}}{c_{11}-f_{20}} = \frac{dc_{11}}{-c_{20}} = \frac{di_{11}}{3j_{11}-i_{20}} = \frac{dj_{11}}{2k_{11}-j_{20}} = \frac{dk_{11}}{l_{11}-k_{20}} = \frac{dl_{11}}{-l_{20}} \\
= &\frac{da_{02}}{-2a_{11}} = \frac{dh_{02}}{g_{02}-2b_{11}} = \frac{dg_{02}}{-2g_{11}} = \frac{db_{02}}{2f_{02}-2b_{11}} = \frac{df_{02}}{c_{02}-2f_{11}} = \frac{dc_{02}}{-2c_{11}} = \frac{di_{02}}{3j_{02}-2i_{11}} = \frac{dj_{02}}{2k_{02}-2j_{11}} = \frac{dk_{02}}{l_{02}-2k_{11}} = \frac{dl_{02}}{-2l_{11}} \\
= &\frac{da_{30}}{0} = \frac{dh_{30}}{g_{30}} = \frac{dg_{30}}{0} = \frac{db_{30}}{2f_{30}} = \frac{df_{30}}{c_{30}} = \frac{dc_{30}}{0} = \frac{di_{30}}{3j_{30}} = \frac{dj_{30}}{2k_{30}} = \frac{dk_{30}}{l_{30}} = \frac{dl_{30}}{0} \\
= &\frac{da_{21}}{-a_{30}} = \frac{dh_{21}}{g_{21}-h_{30}} = \frac{dg_{21}}{-g_{30}} = \frac{db_{21}}{2f_{21}-b_{30}} = \frac{df_{21}}{c_{21}-f_{30}} = \frac{dc_{21}}{-c_{30}} = \frac{di_{21}}{3j_{21}-i_{30}} = \frac{dj_{21}}{2k_{21}-j_{30}} = \frac{dk_{21}}{l_{21}-k_{30}} = \frac{dl_{21}}{-l_{30}} \\
= &\frac{da_{12}}{-2a_{21}} = \frac{dh_{12}}{g_{12}-2h_{21}} = \frac{dg_{12}}{-2g_{21}} = \frac{db_{12}}{2f_{12}-2b_{21}} = \frac{df_{12}}{c_{12}-2f_{21}} = \frac{dc_{12}}{-2c_{21}} = \frac{di_{12}}{3j_{12}-2i_{21}} = \frac{dj_{12}}{2k_{12}-2j_{21}} = \frac{dk_{12}}{l_{12}-2k_{21}} = \frac{dl_{21}}{-2l_{21}} \\
= &\frac{da_{03}}{-3a_{12}} = \frac{dh_{03}}{g_{03}-3h_{12}} = \frac{dg_{03}}{-3g_{12}} = \frac{db_{03}}{2f_{03}-3b_{12}} = \frac{df_{03}}{c_{03}-3f_{12}} = \frac{dc_{03}}{-3c_{12}} = \frac{di_{03}}{3j_{03}-3i_{12}} = \frac{dj_{03}}{2k_{03}-3j_{12}} = \frac{dk_{03}}{l_{03}-3k_{12}} = \frac{dl_{03}}{-3l_{12}}
\end{aligned}
$$

being ninety-nine in all; and therefore ninety-nine independent integrals need to be obtained.

The characteristic equation $D_6 = \Delta = 0$ has the operator Δ given by

$$
\begin{array}{llll}
 & + h_{00}\frac{\partial}{\partial g_{00}} & & + b_{00}\frac{\partial}{\partial f_{00}} \\
- a_{01}\frac{\partial}{\partial a_{10}} & - h_{01}\frac{\partial}{\partial h_{10}} + (h_{10} - g_{01})\frac{\partial}{\partial g_{10}} & - b_{01}\frac{\partial}{\partial b_{10}} & + (b_{10} - f_{01})\frac{\partial}{\partial f_{10}} \\
 & + h_{01}\frac{\partial}{\partial g_{01}} & & + b_{01}\frac{\partial}{\partial f_{01}} \\
-2a_{11}\frac{\partial}{\partial a_{20}} & - 2h_{11}\frac{\partial}{\partial h_{20}} + (h_{20} - 2g_{11})\frac{\partial}{\partial g_{20}} & - 2b_{11}\frac{\partial}{\partial b_{20}} & + (b_{20} - 2f_{11})\frac{\partial}{\partial f_{20}} \\
- a_{02}\frac{\partial}{\partial a_{11}} & - h_{02}\frac{\partial}{\partial h_{11}} + (h_{11} - g_{02})\frac{\partial}{\partial g_{11}} & - b_{02}\frac{\partial}{\partial b_{11}} & + (b_{11} - f_{02})\frac{\partial}{\partial f_{11}} \\
 & + h_{02}\frac{\partial}{\partial g_{02}} & & + b_{02}\frac{\partial}{\partial f_{02}} \\
-3a_{21}\frac{\partial}{\partial a_{30}} & - 3h_{21}\frac{\partial}{\partial h_{30}} + (h_{30} - 3g_{21})\frac{\partial}{\partial g_{30}} & - 3b_{21}\frac{\partial}{\partial b_{30}} & + (b_{30} - 3f_{21})\frac{\partial}{\partial f_{30}} \\
-2a_{12}\frac{\partial}{\partial a_{21}} & - 2h_{12}\frac{\partial}{\partial h_{21}} + (h_{21} - 2g_{12})\frac{\partial}{\partial g_{21}} & - 2b_{12}\frac{\partial}{\partial b_{21}} & + (b_{21} - 2f_{12})\frac{\partial}{\partial f_{21}} \\
- a_{03}\frac{\partial}{\partial a_{12}} & - h_{03}\frac{\partial}{\partial h_{12}} + (h_{12} - g_{03})\frac{\partial}{\partial g_{12}} & - b_{03}\frac{\partial}{\partial b_{12}} & + (b_{12} - f_{03})\frac{\partial}{\partial f_{12}} \\
 & + h_{03}\frac{\partial}{\partial g_{03}} & & + b_{03}\frac{\partial}{\partial f_{03}}
\end{array}
$$

$$
\begin{array}{llll}
+ 2f_{00}\frac{\partial}{\partial c_{00}} & + i_{00}\frac{\partial}{\partial j_{00}} & + 2j_{00}\frac{\partial}{\partial k_{00}} & + 3k_{00}\frac{\partial}{\partial l_{00}} \\
+ (2f_{10} - c_{01})\frac{\partial}{\partial c_{10}} - i_{01}\frac{\partial}{\partial i_{10}} & + (i_{10} - j_{01})\frac{\partial}{\partial j_{10}} & + (2j_{10} - k_{01})\frac{\partial}{d k_{10}} & + (3k_{10} - l_{01})\frac{\partial}{\partial l_{10}} \\
+ 2f_{01}\frac{\partial}{\partial c_{01}} & + i_{01}\frac{\partial}{\partial j_{01}} & + 2j_{01}\frac{\partial}{\partial k_{01}} & + 3k_{01}\frac{\partial}{\partial l_{01}} \\
+ (2f_{20} - 2c_{11})\frac{\partial}{\partial c_{20}} - 2i_{11}\frac{\partial}{\partial i_{20}} & + (i_{20} - 2j_{11})\frac{\partial}{\partial j_{20}} & + (2j_{20} - 2k_{11})\frac{\partial}{\partial k_{20}} & + (3k_{20} - 2l_{11})\frac{\partial}{\partial l_{20}} \\
+ (2f_{11} - c_{02})\frac{\partial}{\partial c_{11}} - i_{02}\frac{\partial}{\partial i_{11}} & + (i_{11} - j_{02})\frac{\partial}{\partial j_{11}} & + (2j_{11} - k_{02})\frac{\partial}{\partial k_{11}} & + (3k_{11} - l_{02})\frac{\partial}{\partial l_{11}} \\
+ 2f_{02}\frac{\partial}{\partial c_{02}} & + i_{02}\frac{\partial}{\partial j_{02}} & + 2j_{02}\frac{\partial}{\partial k_{02}} & + 3k_{02}\frac{\partial}{\partial l_{02}} \\
+ (2f_{30} - 3c_{21})\frac{\partial}{\partial c_{30}} - 3i_{21}\frac{\partial}{\partial i_{30}} & + (i_{30} - 3j_{21})\frac{\partial}{\partial j_{30}} & + (2j_{30} - 2k_{21})\frac{\partial}{\partial k_{30}} & + (3k_{30} - 3l_{21})\frac{\partial}{\partial l_{30}} \\
+ (2f_{21} - 2c_{12})\frac{\partial}{\partial c_{21}} - 2i_{12}\frac{\partial}{\partial i_{21}} & + (i_{21} - 2j_{12})\frac{\partial}{\partial j_{21}} & + (2j_{21} - 2k_{12})\frac{\partial}{\partial k_{21}} & + (3k_{21} - 2l_{12})\frac{\partial}{\partial l_{21}} \\
+ (2f_{12} - c_{03})\frac{\partial}{\partial c_{12}} - i_{03}\frac{\partial}{\partial i_{12}} & + (i_{12} - j_{03})\frac{\partial}{\partial j_{12}} & + (2j_{12} - k_{03})\frac{\partial}{\partial k_{12}} & + (3k_{12} - l_{03})\frac{\partial}{\partial l_{12}} \\
+ 2f_{03}\frac{\partial}{\partial c_{03}} & + i_{03}\frac{\partial}{\partial j_{03}} & + 2j_{03}\frac{\partial}{\partial k_{03}} & + 3k_{03}\frac{\partial}{\partial l_{03}}
\end{array}
$$

91. Proceeding in the usual manner and forming the modified Δ-equations in solutions of the subsidiary equations of $D_1 = 0$, we first take a_{10} for variable of reference. The notation for these solutions of the subsidiary equations will be maintained as in the last case, viz. a solution of $D_1 = 0$ but not of $\Delta = 0$ will be denoted by θ, and one which is simultaneously a solution of $D_1 = 0$ and $D = 0$ arising in the modification of the Δ-equations will be denoted by y.

The quantities y which thus arise are:

$$
\begin{aligned}
y_1 &= a_{00}, \\
y_2 &= (g_{00}, h_{00} \between a_{01}, a_{10}), \\
y_3 &= (c_{00}, f_{00}, b_{00} \between a_{01}, a_{10})^2, \\
y_{14} &= (l_{00}, k_{00}, j_{00}, i_{00} \between a_{01}, a_{10})^3, \\
y_4 &= (g_{10}, h_{10} - g_{01}, -h_{01} \between a_{01}, a_{10})^2, \\
y_5 &= (c_{10}, 2f_{10} - c_{01}, b_{10} - 2f_{01}, -b_{01} \between a_{01}, a_{10})^3, \\
y_{15} &= (l_{10}, 3k_{10} - l_{01}, 3j_{10} - 3k_{01}, i_{10} - 3j_{01}, -i_{01} \between a_{01}, a_{10})^4, \\
y_6 &= (a_{20}, -2a_{11}, a_{02} \between a_{01}, a_{10})^2, \\
y_7 &= (g_{20}, -2g_{11} + h_{20}, g_{02} - 2h_{11}, h_{02} \between a_{01}, a_{10})^3, \\
y_8 &= (c_{20}, -2c_{11} + 2f_{20}, c_{02} - 4f_{11} + b_{20}, 2f_{02} - 2b_{11}, b_{02} \between a_{01}, a_{10})^4, \\
y_{16} &= (l_{20}, -2l_{11} + 3k_{20}, l_{02} - 6k_{11} + 3j_{20}, 3k_{02} - 6j_{11} + i_{20}, 3j_{02} - 2i_{11}, i_{02} \between a_{01}, a_{10})^5, \\
y_{17} &= (a_{30}, -3a_{21}, 3a_{12}, -a_{03} \between a_{01}, a_{10})^3, \\
y_{18} &= (g_{30}, -3g_{21} + h_{30}, 3g_{12} - 3h_{21}, -g_{03} + 3h_{12}, -h_{03} \between a_{01}, a_{10})^4, \\
y_{20} &= (c_{30}, -3c_{21} + 2f_{30}, 3c_{12} - 6f_{21} + b_{30}, -c_{03} + 6f_{12} - 3b_{21}, \\
&\qquad\qquad -2f_{03} + 3b_{12}, -b_{03} \between a_{01}, a_{10})^5, \\
y_{21} &= (l_{30}, -3l_{21} + 3k_{30}, 3l_{12} - 9k_{21} + 3j_{30}, -l_{03} + 9k_{12} - 9j_{21} + i_{30}, \\
&\qquad\qquad -3k_{03} + 9j_{12} - 3i_{21}, -3j_{03} + 3i_{12}, -i_{03} \between a_{01}, a_{10})^6, \\
y_9 &= h_{10} + g_{01}, \\
y_{10} &= (f_{10} + c_{01}, b_{10} + f_{01} \between a_{01}, a_{10}), \\
y_{22} &= (k_{10} + l_{01}, j_{10} + k_{01}, i_{10} + j_{01} \between a_{01}, a_{10})^2, \\
y_{12} &= (h_{20} + g_{11}, -h_{11} - g_{02} \between a_{01}, a_{10}), \\
y_{11} &= (f_{20} + c_{11}, b_{20} - c_{02}, -b_{11} - f_{02} \between a_{01}, a_{10})^2, \\
y_{23} &= (k_{20} + l_{11}, 2j_{20} + k_{11} - l_{02}, -j_{11} + 2k_{02} + i_{20}, -j_{02} - i_{11} \between a_{01}, a_{10})^3, \\
y_{26} &= (h_{30} + g_{21}, -h_{21} - g_{12}, h_{12} + g_{03} \between a_{01}, a_{10})^2, \\
y_{25} &= (f_{30} + c_{21}, b_{30} - f_{21} - 2c_{12}, -2b_{21} - f_{12} + c_{03}, b_{12} + f_{03} \between a_{01}, a_{10})^3, \\
y_{24} &= (k_{30} + l_{21}, 2j_{30} - 2l_{12}, i_{30} - 3j_{21} - 3k_{12} + l_{03}, -2i_{21} + 2k_{03}, i_{12} + j_{03} \between a_{01}, a_{10})^4, \\
y_{13} &= b_{20} + 2f_{11} + c_{02}, \\
y_{27} &= (b_{30} + 2f_{21} + c_{12}, -b_{21} - 2f_{12} - c_{03} \between a_{01}, a_{10}), \\
y_{28} &= (j_{20} + 2k_{11} + l_{02}, i_{20} + 2j_{11} + k_{02} \between a_{01}, a_{10}), \\
y_{29} &= (j_{30} + 2k_{21} + l_{12}, i_{30} + j_{21} - k_{12} - l_{03}, -i_{21} - 2j_{12} - k_{03} \between a_{01}, a_{10})^2, \\
y_{19} &= i_{30} + 3j_{21} + 3k_{12} + l_{03}.
\end{aligned}
$$

The quantities θ which arise in the formation of the modified Δ-equations can be expressed in the following manner: Let any of the quantities y, which

are evidently binary quantics in a_{01}, a_{10}, be of degree r in those two variables; and let the operator

$$\frac{1}{r}\frac{\partial}{\partial a_{01}}$$

be denoted by δ so that the highest power in the derivative has a numerical coefficient unity. Conformably with this definition we have

$$\delta^2 y = \delta . \delta y = \frac{1}{r-1}\frac{\partial}{\partial a_{01}}(\delta y) = \frac{1}{r(r-1)}\frac{\partial^2 y}{\partial a_{01}^2},$$

and so on. Then the quantities θ are:

$$\begin{aligned}
\theta_2 &= a_{10} \quad \text{(the variable of reference)},\\
\theta_3 &= \delta y_2\,,\\
\theta_5\,,\ \theta_4 &= \delta y_3\,,\ \delta^2 y_3\,,\\
\theta_7\,,\ \theta_6 &= \delta y_4\,,\ \delta^2 y_4\,,\\
\theta_{10},\ \theta_9\,,\ \theta_8 &= \delta y_5\,,\ \delta^2 y_5\,,\ \delta^3 y_5\,,\\
\theta_{12},\ \theta_{11} &= \delta y_6\,,\ \delta^2 y_6\,,\\
\theta_{15},\ \theta_{14},\ \theta_{13} &= \delta y_7\,,\ \delta^2 y_7\,,\ \delta^3 y_7\,,\\
\theta_{19},\ \theta_{18},\ \theta_{17},\ \theta_{16} &= \delta y_8\,,\ \delta^2 y_8\,,\ \delta^3 y_8\,,\ \delta^4 y_8\,,\\
\theta_{20} &= \delta y_{10},\\
\theta_{22},\ \theta_{21} &= \delta y_{11},\ \delta^2 y_{11},\\
\theta_{23} &= \delta y_{12},\\
\theta_{26},\ \theta_{25},\ \theta_{24} &= \delta y_{14},\ \delta^2 y_{14},\ \delta^3 y_{14},\\
\theta_{30},\ \theta_{29},\ \theta_{28},\ \theta_{27} &= \delta y_{15},\ \delta^2 y_{15},\ \delta^3 y_{15},\ \delta^4 y_{15},\\
\theta_{35},\ \theta_{34},\ \theta_{33},\ \theta_{32},\ \theta_{31} &= \delta y_{16},\ \delta^2 y_{16},\ \delta^3 y_{16},\ \delta^4 y_{16},\ \delta^5 y_{16},\\
\theta_{38},\ \theta_{37},\ \theta_{36} &= \delta y_{17},\ \delta^2 y_{17},\ \delta^3 y_{17},\\
\theta_{42},\ \theta_{41},\ \theta_{40},\ \theta_{39} &= \delta y_{18},\ \delta^2 y_{18},\ \delta^3 y_{18},\ \delta^4 y_{18},\\
\theta_{47},\ \theta_{46},\ \theta_{45},\ \theta_{44},\ \theta_{43} &= \delta y_{20},\ \delta^2 y_{20},\ \delta^3 y_{20},\ \delta^4 y_{20},\ \delta^5 y_{20},\\
\theta_{53},\ \theta_{52},\ \theta_{51},\ \theta_{50},\ \theta_{49},\ \theta_{48} &= \delta y_{21},\ \delta^2 y_{21},\ \delta^3 y_{21},\ \delta^4 y_{21},\ \delta^5 y_{21},\ \delta^6 y_{21},\\
\theta_{55},\ \theta_{54} &= \delta y_{22},\ \delta^2 y_{22},\\
\theta_{58},\ \theta_{57},\ \theta_{56} &= \delta y_{23},\ \delta^2 y_{23},\ \delta^3 y_{23},\\
\theta_{62},\ \theta_{61},\ \theta_{60},\ \theta_{59} &= \delta y_{24},\ \delta^2 y_{24},\ \delta^3 y_{24},\ \delta^4 y_{24},\\
\theta_{65},\ \theta_{64},\ \theta_{63} &= \delta y_{25},\ \delta^2 y_{25},\ \delta^3 y_{25},\\
\theta_{67},\ \theta_{66} &= \delta y_{26},\ \delta^2 y_{26},\\
\theta_{68} &= \delta y_{27},\\
\theta_{69} &= \delta y_{28},\\
\theta_{71},\ \theta_{70} &= \delta y_{29},\ \delta^2 y_{29},
\end{aligned}$$

wherein the first member on the left-hand side is equal to the first on the right-hand side, the second to the second, and so on.

92. These 70 quantities θ and 29 quantities y make up the necessary number of 99 integrals of the equations subsidiary to $D_1 = 0$. The 99 integrals, as taken above, are independent of one another.

93. In order to express in a concise form the modified Δ-equations, such a relation as

$$\theta_2\Delta\theta_{16} - 4\theta_{16}\Delta\theta_2 = 4\theta_{17}$$

will be represented by

$$[\theta_{16},\, 4] = 4\theta_{17};$$

with this notation the modified Δ-equations are:

$$[\theta_3,\, 1] = y_2 \quad \ldots\ldots \quad \text{(ii)}$$

$$\left.\begin{array}{l} [\theta_4,\, 2] = 2\theta_5 \\ [\theta_5,\, 1] = y_3 \end{array}\right\} \ldots \text{(iii)}$$

$$\left.\begin{array}{l} [\theta_6,\, 2] = 2\theta_7 \\ [\theta_7,\, 1] = y_4 \end{array}\right\} \ldots \text{(iv)}$$

$$\left.\begin{array}{l} [\theta_8,\, 3] = 3\theta_9 \\ [\theta_9,\, 2] = 2\theta_{10} \\ [\theta_{10},\, 1] = y_5 \end{array}\right\} \ldots \text{(v)}$$

$$\left.\begin{array}{l} [\theta_{11},\, 2] = 2\theta_{12} \\ [\theta_{12},\, 1] = y_6 \end{array}\right\} \ldots \text{(vi)}$$

$$\left.\begin{array}{l} [\theta_{13},\, 3] = 3\theta_{14} \\ [\theta_{14},\, 2] = 2\theta_{15} \\ [\theta_{15},\, 1] = y_7 \end{array}\right\} \ldots \text{(vii)}$$

$$\left.\begin{array}{l} [\theta_{16},\, 4] = 4\theta_{17} \\ [\theta_{17},\, 3] = 3\theta_{18} \\ [\theta_{18},\, 2] = 2\theta_{19} \\ [\theta_{19},\, 1] = y_8 \end{array}\right\} \ldots \text{(viii)}$$

$$[\theta_{20},\, 1] = y_{10} \quad \ldots\ldots \quad \text{(x)}$$

$$\left.\begin{array}{l} [\theta_{21},\, 2] = 2\theta_{22} \\ [\theta_{22},\, 1] = y_{11} \end{array}\right\} \ldots \text{(xi)}$$

$$[\theta_{23},\, 1] = y_{12} \quad \ldots\ldots \text{(xii)}$$

$$\left.\begin{array}{l} [\theta_{24},\, 3] = 3\theta_{25} \\ [\theta_{25},\, 2] = 2\theta_{26} \\ [\theta_{26},\, 1] = y_{14} \end{array}\right\} \ldots \text{(xiv)}$$

$$\left.\begin{array}{l} [\theta_{27},\, 4] = 4\theta_{28} \\ [\theta_{28},\, 3] = 3\theta_{29} \\ [\theta_{29},\, 2] = 2\theta_{30} \\ [\theta_{30},\, 1] = y_{15} \end{array}\right\} \ldots \text{(xv)}$$

$$\left.\begin{array}{l} [\theta_{31},\, 5] = 5\theta_{32} \\ [\theta_{32},\, 4] = 4\theta_{33} \\ [\theta_{33},\, 3] = 3\theta_{34} \\ [\theta_{34},\, 2] = 2\theta_{35} \\ [\theta_{35},\, 1] = y_{16} \end{array}\right\} \ldots \text{(xvi)}$$

$$\left.\begin{array}{l} [\theta_{36},\, 3] = 3\theta_{37} \\ [\theta_{37},\, 2] = 2\theta_{38} \\ [\theta_{38},\, 1] = y_{17} \end{array}\right\} \ldots \text{(xvii)}$$

$$\left.\begin{array}{l} [\theta_{39},\, 4] = 4\theta_{40} \\ [\theta_{40},\, 3] = 3\theta_{41} \\ [\theta_{41},\, 2] = 2\theta_{42} \\ [\theta_{42},\, 1] = y_{18} \end{array}\right\} \ldots \text{(xviii)}$$

$$\left.\begin{array}{l} [\theta_{43},\, 5] = 5\theta_{44} \\ [\theta_{44},\, 4] = 4\theta_{45} \\ [\theta_{45},\, 3] = 3\theta_{46} \\ [\theta_{46},\, 2] = 2\theta_{47} \\ [\theta_{47},\, 1] = y_{20} \end{array}\right\} \ldots \text{(xx)}$$

$$\left.\begin{array}{l} [\theta_{48},\, 6] = 6\theta_{49} \\ [\theta_{49},\, 5] = 5\theta_{50} \\ [\theta_{50},\, 4] = 4\theta_{51} \\ [\theta_{51},\, 3] = 3\theta_{52} \\ [\theta_{52},\, 2] = 2\theta_{53} \\ [\theta_{53},\, 1] = y_{21} \end{array}\right\} \ldots \text{(xxi)}$$

$$\left.\begin{array}{l} [\theta_{54},\, 2] = 2\theta_{55} \\ [\theta_{55},\, 1] = y_{22} \end{array}\right\} \ldots \text{(xxii)}$$

$$\left.\begin{array}{l} [\theta_{56},\, 3] = 3\theta_{57} \\ [\theta_{57},\, 2] = 2\theta_{58} \\ [\theta_{58},\, 1] = y_{23} \end{array}\right\} \ldots \text{(xxiii)}$$

$$\left.\begin{array}{l} [\theta_{59},\, 4] = 4\theta_{60} \\ [\theta_{60},\, 3] = 3\theta_{61} \\ [\theta_{61},\, 2] = 2\theta_{62} \\ [\theta_{62},\, 1] = y_{24} \end{array}\right\} \ldots \text{(xxiv)}$$

$$\left.\begin{array}{l} [\theta_{63},\, 3] = 3\theta_{64} \\ [\theta_{64},\, 2] = 2\theta_{65} \\ [\theta_{65},\, 1] = y_{25} \end{array}\right\} \ldots \text{(xxv)}$$

$$\left.\begin{array}{l} [\theta_{66},\, 2] = 2\theta_{67} \\ [\theta_{67},\, 1] = y_{26} \end{array}\right\} \ldots \text{(xxvi)}$$

$$[\theta_{68},\, 1] = y_{27} \ldots \text{(xxvii)}$$

$$[\theta_{69},\, 1] = y_{28} \ldots \text{(xxviii)}$$

$$\left.\begin{array}{l} [\theta_{70},\, 2] = 2\theta_{71} \\ [\theta_{71},\, 1] = y_{29} \end{array}\right\} \ldots \text{(xxix)}$$

Of these 69 equations 68 integrals are required, which, with the 29 simultaneous y-solutions already obtained, make up the (§18) 35 requisite number of $\left(\frac{1}{4}.4.5.4.5 - 3 =\right)$ 97 independent solutions.

94. Now regarding the indicated 25 groups of the complete system of equations, we find first that each group furnishes a certain number of solutions independent of one another and derivable only from that group; and the number of

solutions so furnished is less by unity than the number of equations contained in the group. In fact each group of itself determines the algebraically independent concomitants of the quantity y occurring in the group regarded as a binary quantic; the aggregate of these independent concomitants, excluding the quantity y, will be called the *binariant-system* of that binary quantic y. Thus the binariant-system of

$$u = (A_0, A_1, A_2, \ldots \between x_1, x_2)^n = a_x^n$$

is composed of

$$\frac{1}{2}(ab)^2 a_x^{n-2} b_x^{n-2},$$

$$\frac{1}{2}(ab)^2(ac)\, a_x^{n-3} b_x^{n-2} c_x^{n-1},$$

$$\frac{1}{2}(ab)^4 a_x^{n-4} b_x^{n-4},$$

$$\frac{1}{2}(ab)^4(ac)\, a_x^{n-5} b_x^{n-4} c_x^{n-1},$$

$$\frac{1}{2}(ab)^6 a_x^{n-6} b_x^{n-6},$$

and so on; in terms of these all the invariants and covariants of u can be algebraically expressed.

We thus have a number of solutions for each of the binary quantics y, and derivable from them in the case when y is of degree in a_{01} higher than unity; the number of *additional* solutions thus obtained is

1 from each of the set	$y_3, y_4, y_6, y_{11}, y_{22}, y_{26}, y_{29}$	$= 7$,	
2 " " " "	$y_5, y_7, y_{14}, y_{17}, y_{23}, y_{25}$	$= 12$,	
3 " " " "	$y_8, y_{15}, y_{18}, y_{24}$	$= 12$,	
4 " " " "	y_{16}, y_{20}	$= 8$,	
5 from	y_{21}	$= 5$,	$= 44$ in all.

As we have now used the equations in a group among themselves, we may now take only a single equation out of each group; it is most convenient for the purposes of integration to retain the last equation of the group.

95. We thus have 25 equations left, which will furnish 24 independent integrals.

The combination of any pair of equations leads to the Jacobian of the two quantities y occurring in those equations, regarded as binary quantics in a_{01}, a_{10}; thus from

$$[\theta_3, 1] = y_2, \quad [\theta_5, 1] = y_3$$

we derive a solution

$$\frac{1}{\theta_3}(\theta_5 y_3 - \theta_5 y_2)$$

easily seen to be the Jacobian of y_2 and y_3. Combining, then, equation (ii) in

turn with each of the equations last in the other groups, we have the necessary number of 24 independent solutions; and these are the 24 Jacobians of y_3 and each of the remaining quantities y which are not independent of a_{01} and a_{10}. Combining, then, all our solutions, we have

(i) the 29 quantities y,
(ii) the 44 derived through the binariant systems,
(iii) the 24 Jacobians,

making the total of 97, the required number.

96. The process of derivation from the 99 independent solutions of $D_1 = 0$ shows that the 97 simultaneous solutions are independent of one another; it follows from the theory that every simultaneous solution can be expressed in terms of them.

The order and class will be left undetermined until §§106–112, when they will be given for the general quantic.

97. If we take $\theta_3 = g_{00}$ as the variable of reference instead of $\theta_2 = a_{10}$ and proceed in the same way, we find a set of binary quantics which have $-h_{00}$ and $+g_{00}$ for their variables instead of a_{01} and a_{10}. The forms $y_1, y_2, y_9, y_{13}, y_{19}$ are unaltered; the remainder have their coefficients the same, and their modification consists in the mentioned change of variables.

The aggregate of independent simultaneous solutions is similarly constituted; we have in addition to the quantics their binariant systems, and the set of Jacobians taken of course with regard to the variables of the system of quantics.

We shall denote the quantics in these variables by z, so that if $y_\mu = (*\between a_0{}^1, a_{10})^\lambda$ for any index μ and degree λ, then z_μ will denote $(*\between -h_{00}, g_{00})^\lambda$ with the same coefficients as y_μ.

VI.—*The Ternary Quantic of Order n and Class m.*

98. The complete system of algebraically independent concomitants consists of three classes, the arrangement being made conveniently with regard to their leading coefficients.

The first class of leading coefficients consists of a number of binary quantics in a_{01} and a_{10} as variables; (the results will be enunciated only for this system, but it may be borne in mind that there is an equivalent system in g_{00} and $-h_{00}$ as variables).

The second class of leading coefficients is constituted by the several bina-

riant systems of each of the binary quantics in the first class taken singly; that is to say, the system of algebraically independent concomitants of any such quantic, the quantic itself excluded. It is evident that any quantic of the first degree only in a_{01} and a_{10}, or one of zero degree in them (that is to say, independent of them) will supply no leading coefficients to this class.

The third class of leading coefficients is constituted by the Jacobians of any one quantic which involves a_{01} and a_{10} with each of the others in turn which also involve these variables.

It thus appears that, if the first class be completely given, then the second and third classes can be derived from them.

99. Let y be any one of the leading coefficients of the first class, determining a concomitant of the form

$$yx_1^\mu u_1^\nu + \ldots .$$

A linear substitution is $x_1 = X_1$, $x_2 = X_3$, $x_3 = X_2$, which must leave the concomitant unchanged (save possibly as to sign), and must therefore leave y similarly unchanged. The effect of this substitution is to interchange coefficients of the quantic symmetrically associated with x_2 and x_3, u_2 and u_3; this interchange must therefore not affect y, a binary quantic in a_{01} and a_{10}. But a_{01} and a_{10} are interchanged by the substitution; hence the sole effect on y (except a possible change of sign) is to reverse the order of the terms.

Let $a_{01}^p A$ be the first term in y; the form of y is

$$a_{01}^p A + a_{01}^{p-1} a_{10} \Delta A + \frac{1}{2!} a_{01}^{p-2} a_{10}^2 \Delta^2 A + \ldots .$$

as follows from the differential equation $D_6 = \Delta = 0$ to be satisfied by y. The last term in the series will be $a_{10}^p A'$, where A' is the value of A when the above interchange is effected.

It thus appears that a knowledge of the single term $a_{01}^p A$ is sufficient to determine y. But now we proceed to show, what is indeed the ordinary inference in the theory of binary quantics, that a knowledge of A alone is sufficient to determine y.

For y is isobaric and therefore $a_{01}^p A$ and $a_{10}^p A'$ are of the same weight. Denoting the quantic by $a_x^n u_\alpha^m$ and using the assignation of weights in §4, we have

$$p = \frac{\text{weight of } A - \text{weight of } A'}{\text{weight of } a_{10} - \text{weight of } a_{01}}.$$

But the umbral values of a_{01} and a_{10} are $a_1^n \alpha_1^{m-1} \alpha_3$ and $a_1^n \alpha_1^{m-1} \alpha_2$ respectively, so that

$$\text{weight of } a_{10} - \text{weight of } a_{01} = \text{weight of } \alpha_2 - \text{weight of } \alpha_3$$
$$= 1,$$

and therefore $\qquad p = \text{weight of } A - \text{weight of } A'.$

Thus when A is known we can deduce p and so find y; and for this purpose it is really sufficient to take any term in A, obtain the corresponding term in A' by the interchange of coefficients of the quantic symmetrical with regard to u_2 and u_3, x_2 and x_3, and find the difference of the weights which determines p. For instance, in the case of the cubo-cubic we have in y_{24} as coefficient of the first term $k_{30} + l_{21}$; taking k_{30}, which is the coefficient of $x_2 x_3^2 u_2^3$ (disregarding numerical coefficients), we change it into the coefficient of $x_3 x_2^2 u_3^3$, i. e. into j_{03}, so that

$$\begin{aligned} p &= \text{weight of } k_{30} - \text{weight of } j_{03} \\ &= \text{weight of } a_2 a_3^2 \alpha_2^3 - \text{weight of } a_2^2 a_3 \alpha_3^3 \\ &= \text{weight of } a_3 - \text{weight of } a_2 + 3\,(\text{weight of } \alpha_2 - \text{weight of } \alpha_3) \\ &= 1 + 3.1 = 4, \end{aligned}$$

agreeing with the form there given.

Hence it appears that the theory of binary quantics applies, so that *if we know the coefficient of the highest power of* a_{01} *in* y, *and even nothing but this coefficient, we can obtain the value of* y *by pure differentiation with the operator* Δ.

The determination of the quantics y therefore resolves itself into a determination of the coefficients A of their first terms.

100. In the general biternary quantic we write $(s,\,t)_{\sigma,\,\tau}$ in place of $a_{r,\,s,\,t,\,\rho,\,\sigma,\,\tau}$ (with the conditions $n = r + s + t$, $m = \rho + \sigma + \tau$); so that $(s,\,t)_{\sigma,\,\tau}$ is the literal coefficient of $x_1^r x_2^s x_3^t u_1^\rho u_2^\sigma u_3^\tau$ and its umbral value is given by

$$(s,\,t)_{\sigma,\,\tau} = a_1^{n-s-t} a_2^s a_3^t \alpha_1^{m-\sigma-\tau} \alpha_2^\sigma \alpha_3^\tau.$$

The operator $\Delta\,(= D_6)$ of §59 is in this notation

$$\sum \left[\{ t\,(s+1,\,t-1)_{\sigma,\,\tau} - \sigma\,(s,\,t)_{\sigma-1,\,\tau+1} \} \frac{\partial}{\partial\,(s,\,t)_{\sigma,\,\tau}} \right],$$

the summation extending over all the values of s and t such that $s + t$ is not greater than n, and all values of σ and τ such that $\sigma + \tau$ is not greater than m.

101. *All the leading coefficients A of the binary quantics y, which are themselves leading coefficients of concomitants, are included in the formula*

$$(0,\,t)_{\tau-\lambda,\,\lambda} + \lambda\,(1,\,t-1)_{\tau-\lambda+1,\,\lambda-1} + \frac{\lambda\,(\lambda-1)}{2!}\,(2,\,t-2)_{\tau-\lambda+2,\,\lambda-2} + \ldots$$
$$+ \frac{\lambda\,(\lambda-1)}{2!}\,(\lambda-2,\,t-\lambda+2)_{\tau-2,\,2} + \lambda\,(\lambda-1,\,t-\lambda+1)_{\tau-1,\,1} + (\lambda,\,t-\lambda)_{\tau,\,0}$$

for the values $t = \lambda, \lambda + 1, \ldots, n,$
$\tau = \lambda, \lambda + 1, \ldots, m$

for each single value of λ; *and the values of* λ *are*

$$\lambda = 0, 1, 2, \ldots, m \text{ or } n$$

according as m is less or is greater than n. Such a quantity we shall represent by $A_{t,\tau,\lambda}$.

Having now the coefficient of the first term in the quantic y, it is necessary to determine the degree p of that quantic in a_{01}. Taking any term of $A_{t,\tau,\lambda}$, say the first which is $(0, t)_{\tau-\lambda,\lambda}$, we make the substitution which interchanges the terms in x_2 and x_3, u_2 and u_3, before indicated; this interchange gives us $(t, 0)_{\lambda,\tau-\lambda}$, so that

$$\begin{aligned} p &= \text{weight of } (0, t)_{\tau-\lambda,\lambda} - \text{weight of } (t, 0)_{\lambda,\tau-\lambda} \\ &= \text{weight of } a_1^{n-t}a_3^t\alpha_1^{m-\tau}\alpha_2^{\tau-\lambda}\alpha_3^{\lambda} - \text{weight of } a_1^{n-t}a_2^t\alpha_1^{m-\tau}\alpha_2^{\lambda}\alpha_3^{\tau-\lambda} \\ &= t(\text{weight of } a_3 - \text{weight of } a_2) + (\tau - 2\lambda)(\text{weight of } \alpha_2 - \text{weight of } \alpha_3) \\ &= t + \tau - 2\lambda. \end{aligned}$$

Hence *the quantic, which may be denoted by* $y_{t,\tau,\lambda}$, is

$$y_{t,\tau,\lambda} = \left(\left\{1, \Delta, \frac{1}{2!}\Delta^2, \frac{1}{3!}\Delta^3, \ldots, \frac{1}{t+\tau-2\lambda!}\Delta^{t+\tau-2\lambda}\right\} A_{t,\tau,\lambda} \between a_{01}, a_{10}\right)^{t+\tau-2\lambda},$$

with the foregoing limitations on the values of t, τ, λ; *the quantities* a_{01} *and* a_{10} *denoting* $(0, 0)_{0,1}$ *and* $(0, 0)_{1,0}$.

For instance, in the case of the cubo-cubic the several quantics

	t	τ	λ		t	τ	λ		t	τ	λ		t	τ	λ		t	τ	λ
y_1	0	0	0	y_5	2	1	0	y_{17}	0	3	0	y_{23}	3	1	1	y_{24}	3	3	1
y_2	1	0	0	y_{15}	3	1	0	y_{18}	1	3	0	y_{12}	1	2	1	y_{13}	2	2	2
y_3	2	0	0	y_6	0	2	0	y_{20}	2	3	0	y_{11}	2	2	1	y_{28}	3	2	2
y_{14}	3	0	0	y_7	1	2	0	y_{21}	3	3	0	y_{22}	3	2	1	y_{27}	2	3	2
				y_8	2	2	0	y_9	1	1	1	y_{26}	1	3	1	y_{29}	3	3	2
y_4	1	1	0	y_{16}	3	2	0	y_{10}	2	1	1	y_{25}	2	3	1	y_{19}	3	3	3

are given in the accompanying table. The reason that there is no entry here for $t, \tau, \lambda = 0, 1, 0$ is that the corresponding coefficient of the first term is $(0, 0)_{01}$

which is one of the variables of the quantic. And this omission is general for the special group of values.

102. Let N be the total number of quantics in the above system for the $n^o m^{1o}$; then, adding unity on account of the single omission just referred to, we have

$$\begin{aligned} N + 1 &= (m+1)(n+1) \text{ for the value } \lambda = 0 \\ &\quad + mn \qquad\qquad\quad \text{"} \qquad \text{"} \quad \lambda = 1 \\ &\quad + (m-1)(n-1) \quad \text{"} \qquad \text{"} \quad \lambda = 2 \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= (m+1)(n+1) + mn + (m-1)(n-1) + \ldots\ldots, \end{aligned}$$

the series containing either $n+1$ or $m+1$ terms, whichever is the smaller integer.

We thus have *all the leading coefficients of the first class*, and the number of them.

103. The *leading coefficients of the second class* are constituted by the members of the binariant systems of those of the first class considered as isolated binary quantics. The number of members in the binariant system of a binary quantic of degree p is $p-1$, provided p be not less than 2; but if p be zero or unity, there is no binariant system. Hence only those quantics of the previous class for which $t + \tau \geqq 2\lambda + 2$ will furnish members of the second class of coefficients; and, if $y_{t,\tau,\lambda}$ be one such, the number it furnishes is $t + \tau - 2\lambda - 1$. Thus the total number of coefficients of this class is $N' = \Sigma\,(t + \tau - 2\lambda - 1)$ with the limitations

$$\begin{gathered} t + \tau \geqq 2\lambda + 2, \\ t \geqq \lambda \leqq n, \\ \tau \geqq \lambda \leqq m. \end{gathered}$$

104. The *leading coefficients of the third class* are constituted by the Jacobian of any one of the first class involving a_{01} and a_{10} (say $y_{1,0,0}$) with each of the others of the first class involving those quantities; and it is with regard to a_{01} and a_{10} that the Jacobians must be taken. Now, of the N quantics in the first class there are either $n+1$ or $m+1$ (whichever be the smaller integer) which do not involve a_{01} and a_{10}—they are in fact given by $t = \tau = \lambda$—and therefore the number of quantics, other than $y_{1,0,0}$, which do involve those quantities is

$$N'' = N - 1 - \begin{cases} n+1 \\ m+1 \end{cases},$$

taking in the last term the smaller of the two integers. Each such quantic

combined with $y_{1,0,0}$ furnishes a Jacobian, and therefore the number of leading coefficients of the third class is N''.

Hence the *total number of leading coefficients of all classes* is

$$N + N' + N'';$$

and each of these leading coefficients determines one of the system of algebraically independent concomitants of the biternary quantic, in terms of which any concomitant can be expressed.

105. As regards the equivalent system obtained by taking g_{00} as the variable of reference, exactly similar results are obtained. There is a set of N-quantics in $-h_{00}$ and g_{00} as variables and of the same degrees, so that we have

$$z_{t,\tau,\lambda} = \left(\left\{1,\ \Delta,\ \frac{1}{2!}\Delta^2,\ \ldots,\ \frac{1}{t+\tau-2\lambda!}\Delta^{t+\tau-2\lambda}\right\} A_{t,\tau,\lambda} \between -h_{00};\ g_{00}\right)^{t+\tau-2\lambda}.$$

We have N' further coefficients of concomitants obtained by taking the various binariant systems of these z-quantics; and the third class of N'' Jacobians of any one of them, say $z_{1,0,0}$, with all the others, the variables being in the present case $-h_{00}$, g_{00}.

106. Having now obtained the leading coefficients, it is necessary to determine *the order and the class of each of the concomitants* so determined; for this purpose the symbolical method will be adopted.

We first change $A_{t,\tau,\lambda}$ into its symbolical form, which is easily found to be

$$a_1^{n-t}\alpha_1^{m-\tau}a_3^{t-\lambda}\alpha_2^{\tau-\lambda}(a_2\alpha_2 + a_3\alpha_3)^{\lambda}.$$

The effect of the operator Δ on $A_{t,\tau,\lambda}$ is to change $(s,\ t)_{\sigma,\tau}$, that is, $a_1^{n-s-t}a_2^s a_3^t \alpha_1^{m-\sigma-\tau}\alpha_2^{\sigma}\alpha_3^{\tau}$ into

$$t\,(s+1,\ t-1)_{\sigma,\tau} - \sigma\,(s,\ t)_{\sigma-1,\tau+1},$$

that is, into

$$ta_1^{n-s-t}a_2^{s+1}a_3^{t-1}\alpha_1^{m-\sigma-\tau}\alpha_2^{\sigma}\alpha_3^{\tau} - \sigma a_1^{n-s-t}a_2^s a_3^t\alpha_1^{m-\sigma-\tau}\alpha_2^{\sigma-1}\alpha_3^{\tau+1},$$

so that in the symbolical form the effect of the operator Δ is

$$\alpha_2\frac{\partial}{\partial\alpha_3} - \alpha_3\frac{\partial}{\partial\alpha_2},$$

and similarly for repetitions of the operator. Now when this symbolical Δ-form operates on $(a_2\alpha_2 + a_3\alpha_3)$ the result is zero, so that this quantity behaves like a constant for Δ; hence we have

$$\begin{aligned} y_{t,\tau,\lambda} &= a_1^{n-t}\alpha_1^{m-\tau}(a_2\alpha_2 + a_3\alpha_3)^\lambda\left[\left(\left\{1,\ \Delta,\ \frac{\Delta^2}{2!},\ \ldots\right\} a_3^{t-\lambda}\alpha_2^{\tau-\lambda}\,)\!(\, a_{01},\ a_{10}\right)^{t+\tau-2\lambda}\right] \\ &= a_1^{n-t}\alpha_1^{m-\tau}(a_2\alpha_2 + a_3\alpha_3)^\lambda(a_3a_{10} + a_3a_{01})^{t-\lambda}(\alpha_3a_{01} - \alpha_3a_{10})^{\tau-\lambda}, \end{aligned}$$

after substitution and reduction. And in this expression all the symbols except a_{01} and a_{10} are umbral.

107. We can at once derive from this form of $y_{t,\tau,\lambda}$ the order and the class to be associated with it, completing the elements of the concomitant. For every factor of the form $a_2\alpha_2 - a_3\alpha_3$—that is, $a_\alpha - a_1\alpha_1$—there are a single power of x and a single power of u occurring. For every factor of the form $a_2a_{10} + a_3a_{01}$—that is, $b_1^n\beta^{n-1}(a_2\beta_2 + a_3\beta_3) = b_1^n\beta_1^{m-1}(a_\beta - a_1\beta_1)$ there are a power $n+1$ of x and a power m of u occurring. For every factor of the form $\alpha_2a_{01} - \alpha_3a_{10}$—that is, $c_1^n\gamma_1^{m-1}(\alpha_3\gamma_3)$—there are a power $n+1$ of x and a power $m-1$ of u occurring. Hence the order in the x-variables is

$$n - t + \lambda + (t-\lambda)(n+1) + (n+1)(\tau-\lambda) = n(t+\tau-2\lambda) + n - \lambda + \tau;$$

and the class in the u-variables is

$$m - \tau + \lambda + (t-\lambda)\,m + (m-1)(\tau-\lambda) = m(t+\tau-2\lambda) + m - 2\tau + 2\lambda.$$

But by means of concomitants occurring earlier in the sequence, it is possible (as in §79) to take a linear combination of $y_{t,\tau,\lambda}$ and powers and products of those earlier concomitants such that the symbolical form of the concomitant determined by the linear combination is divisible by a power of u_x equal to $\lambda + (t-\lambda)$, i. e. by u_x^t; and thus $y_{t,\tau,\lambda}$ determines a concomitant which may be called congruent with

$$a_\alpha^\lambda a_x^{n-t}u_\alpha^{m-\tau}\prod^{t-\lambda}\{a_\beta b_x^n u_\beta^{m-1}\}\prod^{\tau-\lambda}\{c_x^n u_\gamma^{m-1}(\alpha\gamma x)\}.$$

Retaining, however, the simpler form of leading coefficient, the concomitant thence determined is

$$y_{t,\tau,\lambda}x_1^{n(t+\tau-2\lambda+1)+\tau-\lambda}u_1^{m(t+\tau-2\lambda+1)-2\tau+2\lambda} + \ldots;$$

and thus the order and the class of each concomitant of the first class of leading coefficients are determined.

108. Passing now to the second class of leading coefficients, constituted by the binariant systems of those in the first class, we know that they can be arranged in two sets which are respectively of the second and the third degrees

in the *coefficients* of the binary quantic coefficients, but which have not yet been given in any form either really or umbrally connected with the coefficients of biternary quantic. For this purpose let

$$y_{t,\tau,\lambda} = y_\xi^{p+q} = Aa_\xi^p \theta_\xi^q,$$

where $p = t - \lambda$, $q = \tau - \lambda$, $A = a_1^{n-t} \alpha_1^{m-\tau} (a_2\alpha_2 + a_3\alpha_3)^\lambda$, $a_\xi = a_2\alpha_{10} + a_3\alpha_{01}$, $\theta_\xi = \alpha_2 a_{01} - \alpha_3 a_{10}$. Then the transvectants may be represented in the forms

$$(s)_{t,\tau,\lambda} = (yy')^{2s} y_\xi^{p+q-2s} y_\xi'^{p+q-2s}$$

for those of the second degree in the *coefficients* of y, and in the forms

$$(s')_{t,\tau,\lambda} = (yy')^{2s}(yy'') y_\xi^{p+q-2s-1} y_\xi'^{p+q-2s} y_\xi''^{p+q-1}$$

for those of the third degree in the *coefficients* of y.

109. Consider first the former class, those of the second degree in the coefficients of y. We have

$$y_\xi^{p+q} = Aa_\xi^p \theta_\xi^q, \quad y_\xi'^{p+q} = Bb_\xi^p \phi_\xi^q;$$

so that

$$y_\xi^{p+q-2s}(yy')^{2s} y_\xi'^{p+q-2s} = \Sigma A a_\xi^{p-\rho} \theta_\xi^{q-\sigma} (ay')^\rho (\theta y')^\sigma y_\xi'^{p+q-\rho-\sigma},$$

the numerator on the right-hand side extending to all values of ρ and σ such that $\rho + \sigma = 2s$. Also

$$y_\xi'^{p+q-\rho-\sigma} y_\eta'^{\rho} y_\zeta'^{\sigma} = \Sigma B b_\xi^{p-\nu-\mu} b_\eta^\nu b_\zeta^\mu \phi_\xi^{q-\rho-\sigma+\nu+\mu} \phi_\eta^{\rho-\nu} \phi_\zeta^{\sigma-\mu},$$

the summation on the right-hand side being for values $0, 1, \ldots, \rho$ of ν and values $0, 1, \ldots, \sigma$ of μ. Hence

$$\begin{aligned}(s)_{t,\tau,\lambda} &= (yy')^{2s} y_\xi^{p+q-2s} y_\xi'^{p+q-2s} \\ &= AB\Sigma a_\xi^{p-\rho} \theta_\xi^{q-\sigma} b_\xi^{p-\nu-\mu} (ab)^\nu (a\phi)^{\rho-\nu} (\theta b)^\mu (\theta\phi)^{\sigma-\mu} \phi_\xi^{q-\rho-\sigma+\nu+\mu},\end{aligned}$$

the summation extended to all values $0, 1, \ldots, \rho$ of ν; to all values $0, 1, \ldots, \sigma$ of μ, and to all values of ρ and σ such that $\rho + \sigma = 2s$.

When the various terms in this summation are completed into forms which contain the variables, so as to give the concomitant having $(s)_{t,\tau,\lambda}$ for its leading coefficient, it appears that they are of varying order in x and of varying class in u. But, as will be seen immediately, the difference between the order and the class is the same for all the terms; and therefore, on the multiplication of each

term by a power of u_x proper to the term, all the terms are made to be of the same order throughout and the same class throughout. And evidently this order and this class are the order and the class of the particular term, or aggregate of terms, in the summation, and they give when completed the highest order and the highest class of all the terms.

Considering, then, the term occurring under the sign of summation as the typical term, we have as in §107

the order in x-variables

$$\begin{array}{ll} = 2(n-t) + 2\lambda & \text{from } AB \\ \quad + (p-\rho)(n+1) & \text{from } a_\xi^{p-\rho} \\ \quad + (q-\sigma)(n+1) & \text{from } \theta_\xi^{q-\sigma} \\ \quad + (p-\nu-\mu)(n+1) & \text{from } b_\xi^{p-\nu-\mu} \\ \quad + \rho - \nu & \text{from } (a\phi)^{\rho-\nu} \text{ for } (a\phi) = a_2\phi_2 + a_3\phi_3 = a_\phi - a_1\phi_1 \\ \quad + \mu & \text{from } (\theta b)^\mu \\ \quad + \sigma - \mu & \text{from } (\theta\phi)^{\sigma-\mu} \\ \quad + (q-\rho-\sigma+\nu+\mu)(n+1) & \text{from } \phi_\xi^{q-\rho-\sigma+\nu+\mu} \\ = 2(n-t+\lambda) + 2(n+1)(p+q-\rho-\sigma) + \rho + \sigma - \nu; & \end{array}$$

while from the same typical term the order in u-variables

$$\begin{array}{ll} = 2(m-\tau) + 2\lambda & \text{from } AB \\ \quad + (p-\rho)\,m & \text{from } a_\xi^{p-\rho} \\ \quad + (q-\sigma)(m-1) & \text{from } \theta_\xi^{q-\sigma} \\ \quad + (p-\nu-\mu)\,m & \text{from } b^{p-\nu-\mu} \\ \quad + (q-\rho-\sigma+\nu+\mu)(m-1) & \text{from } \phi_\xi^{q-\rho-\sigma+\nu+\mu} \\ \quad + \nu & \text{from } (ab)^\nu \\ \quad + (\rho-\nu) & \text{from } (a\phi)^{\rho-\nu} \\ \quad + \mu & \text{from } (\theta b)^\mu \\ = 2(m-\tau+\lambda) + 2m(p+q-\rho-\sigma) - 2q + 2(\sigma+\rho) - \nu. & \end{array}$$

The difference of these two is at once seen to depend only upon $n, m; t, \tau, \lambda$; and $\rho + \sigma \; (= 2s)$ and is therefore the same for all terms.

The greatest value of each is given by the terms for which $\nu = 0$, so that the order of the concomitant is

$$2(n-t+\lambda) + 2(n+1)(p+q-2s) + 2s,$$

and its class is

$$2(m-\tau+\lambda)+2m(p+q-2s)-2q+4s;$$

and the power of u_x, which must be associated with the foregoing typical term in its completed form, is u_x^ν. These are *the order and the class of the concomitant having as its leading coefficient* $(s)_{t,\tau,\lambda}$, the transvectant of the second degree and s^{th} rank of $y_{t,\tau,\lambda}$.

110. But, as in §§73 and 79, the preceding concomitant can, by the addition of suitable combinations of concomitants occurring earlier in the series, be reduced so as to leave only that single term which involves the highest power of u_x in the whole sum of terms which is the expression of the concomitant; and the concomitant can therefore be considered as congruent to the function given by that single term when the power of u_x has been removed from it.

Now the highest power of u_x occurring in the completed form of the typical term is

$$\begin{aligned}
&\lambda \text{ from } A, \text{ for } A \text{ gives when completed } a_x^{n-t}u_\alpha^{m-\tau}(a_\alpha u_x - a_x u_\alpha)^\lambda \\
&+\lambda \text{ from } B, \text{ similarly} \\
&+p-\rho \text{ from } a_\xi^{p-\rho} \\
&+p-\nu-\mu \text{ from } b_\xi^{p-\nu-\mu} \\
&+\rho-\nu \text{ from } (a\phi)^{\rho-\nu} \\
&+\mu \text{ from } (\theta b)^\mu \\
&= 2p+2\lambda-2\nu,
\end{aligned}$$

and the term or set of terms for which this is greatest are the terms given by $\nu=0$, so that the power of u_x to be removed is $u_x^{2p+2\lambda}$.

And the function to which the preceding concomitant is thus reduced is the sum of quantities

$$[a_\alpha^\lambda b_\beta^\lambda a_x^{n-t}u_\alpha^{m-\tau}b_x^{n-t}u_\beta^{m-\tau}a_\beta^\rho b_\alpha^\mu(\alpha\beta x)^{\sigma-\mu}]$$
$$[\overset{p-\rho}{\Pi} c_x^n u_\gamma^{m-1}a_\gamma][\overset{q-\sigma}{\Pi} d_x^n u_\delta^{m-1}(\alpha\delta x)][\overset{p-\mu}{\Pi} e_x^n u_\epsilon^{m-1}b_\epsilon][\overset{q-\rho-\sigma+\mu}{\Pi} k_x^n u_\kappa^{m-1}(\beta\kappa x)]$$

for all values of ρ and σ such that $\rho+\sigma=2s$, the symbol Π implying the product of t quantities similar to those which immediately follow that symbol.

111. Similarly proceeding with the concomitant, whose leading coefficient $s')_{t,\tau,\lambda}$ is of the third degree in the coefficients of $y_{t,\tau,\lambda}$, we find that the order of the concomitant is

$$3(n-t+\lambda)+(n+1)(3p+3q-4s-2)+2s+1,$$

and that the class is

$$3(m-\tau+\lambda)+m(3p+3q-4s-2)-3q+4s+2.$$

All the elements of the concomitants, determined by the second class of leading coefficients, have now been obtained.

112. Lastly, for the third class of coefficients constituted by the Jacobians, we take the Jacobians of $y_{1,0,0}$ with all the other quantics. The Jacobian of $y_{1,0,0}$ with $y_{t,\tau,\lambda}$ can be taken, with the preceding notation, in the form

$$\frac{A}{p+q}\, b_1^{n-1}\beta_1^m\{p a_\xi^{p-1}\theta_\xi^q(ab)+q a_\xi^p\theta_\xi^{q-1}(\theta b)\},$$

from which it appears that the order in x is

$$(p+b+1)(n+1)-2-t+\lambda,$$

and that the class in u is

$$(p+q+1)m+2-\tau-q+\lambda.$$

This completes the determination of the elements of all the concomitants in the algebraically independent system of the biternary $n^o m^{io}$.

113. As a special case of the foregoing, serving to render the results obtained more precise, I add the elements of the system of the *quadro-quadratic*, represented by

$$\begin{aligned}
&(a_{00}x_1^2+2h_{00}x_1x_2+2g_{00}x_1x_3+b_{00}x_2^2+2f_{00}x_2x_3+c_{00}x_3^2)\,u_1^2\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
+&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
+&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
+&(a_{02}x_1^2+2h_{02}x_1x_2+2g_{02}x_1x_3+b_{02}x_2^2+2f_{02}x_2x_3+c_{02}x_3^2)\,u_3^2.
\end{aligned}$$

The quantities y_μ are the same as those denoted by the same symbols in §91; the values of t, τ, λ are those to be associated with y_μ from the preceding general investigation; h_μ is the Hessian of y_μ, so that $s=1$ and ϕ_μ is its cubicovariant, for which also $s=1$; i_μ is the quadrinvariant of y_μ for which $s=2$; and $j_{2,\mu}$ is the Jacobian of y_2 and y_μ. The values of m and of p are the orders in x-variables and the classes in u-variables of the concomitants determined by the leading coefficients; and the necessary 33 concomitants (§§18 and 35) of the system for the quadro-quadratic have their elements as given in the following table:

t	τ	λ	FIRST CLASS OF LEADING COEFFICIENT.	m	p	SECOND CLASS OF LEADING COEFFICIENT.	m	p	THIRD CLASS OF LEADING COEFFICIENT.	m	p
0	0	0	y_1	2	2						
1	0	0	y_2	4	4						
2	0	0	y_3	6	6	h_3	2	8	$j_{2,3}$	5	8
1	1	0	y_4	7	4	h_4	4	4	$j_{2,4}$	6	6
2	1	0	y_5	9	6	h_5	8	8	$j_{2,5}$	8	8
						ϕ_5	12	12			
0	2	0	y_6	8	2	h_6	6	0	$j_{2,6}$	7	4
1	2	0	y_7	10	4	h_7	10	4	$j_{2,7}$	9	6
						ϕ_7	15	6			
2	2	0	y_8	12	6	h_8	14	8	$j_{2,8}$	11	8
						ϕ_8	21	12			
						i_8	4	4			
2	2	1	y_{11}	7	4	h_{11}	4	4	$j_{2,11}$	6	6
2	1	1	y_{10}	4	4				$j_{2,10}$	8	6
1	2	1	y_{12}	5	2				$j_{2,12}$	4	4
1	1	1	y_9	2	2						
2	2	2	y_{13}	2	2						

In terms of these concomitants every concomitant of the quadrato-quadratic can be expressed; the simplest cases of all appear to be

$$(Y_1 + Y_9 + Y_{13}) \div u_x^2 = \text{linear invariant } a_\alpha^2,$$

$$(Y_1 + Y_9) \div u_x \quad = \text{linear concomitant } a_\alpha a_x u_\alpha.$$

The following short abstract of the contents of the paper may prove useful for reference:

ERRATA.

P. 4, l. 4, for concomitants read *quantics*.
P. 12, l. 13, for also $\pm$ is read *also is* $\pm$.
P. 31, l. 11, for U_1 read U_0.
P. 32, l. 10, for $\frac{\Phi}{3}$ read Φ_3.

On Some Applications of Circular Coordinates.

BY F. FRANKLIN.

The interesting geometrical questions treated by Humbert in a recent number of this Journal (Vol. X, p. 258) may be investigated with advantage by the use of "circular coordinates." The theorems relating to the orientation of systems of lines given in the article just cited, and the more general theorems of the same nature due to Laguerre and Humbert (see Humbert, Sur le théorème d'Abel et quelques-unes de ses applications géométriques, *Liouville*, 1887, III, 327), present themselves at once; some of them may be stated in a way which suggests more readily certain interesting cases of the theorems; and there naturally arise also some slight additions to the theorems.

A second application of circular coordinates is made in this paper. Namely, it is obvious that when x, y are understood to be circular coordinates, the differential equation

$$\frac{dx}{(x-a_1)^{q_1}(x-a_2)^{q_2}\ldots\ldots(x-a_n)^{q_n}}=\frac{dy}{(y-b_1)^{q_1}(y-b_2)^{q_2}\ldots\ldots(y-b_n)^{q_n}}$$

defines a curve in which the angle made by the tangent with a fixed line is a linear combination of the angles made with that line by the rays drawn from its point of contact to a set of fixed points. Thus the discovery of any curve defined in this way is reduced to a question of quadrature; and on the other hand the integration of the differential equation is accomplished if the curves possessing this property are known. It is obvious, further, that the equation

$$e^{-ia}\frac{dx}{(x-a_1)^{q_1}(x-a_2)^{q_2}\ldots\ldots(x-a_n)^{q_n}}=e^{ia}\frac{dy}{(y-b_1)^{q_1}(y-b_2)^{q_2}\ldots\ldots(y-b_n)^{q_n}}$$

defines an oblique trajectory of the foregoing curves. I give a number of illustrations of this geometrical interpretation of such equations, treating in conclusion the curve defined by the equation

$$\sin x\,dx=\sin y\,dy,$$

which, when $\sin x$ is represented as an infinite product, falls under the above head. In connection with this case, I discuss the values of certain series which happen to be suggested by it, and which seemed, though not specially pertaining to the subject, to be of sufficient interest to warrant their consideration here.

I.—INTRODUCTORY.

If X, Y denote rectangular coordinates, the name of "circular coordinates" has been given to the quantities $X+iY$, $X-iY$, which we shall denote by x, y. Thus

$$x = X + iY = re^{i\vartheta}, \quad y = X - iY = re^{-i\vartheta}, \tag{1}$$

where r, ϑ are the ordinary polar coordinates of the point (X, Y); and hence

$$\frac{x}{y} = e^{2i\vartheta}. \tag{2}$$

If ϑ denote the angle made with $Y=0$ by the line joining the points 1 and 2, we have

$$e^{2i\vartheta} = \frac{x_2 - x_1}{y_2 - y_1}, \tag{3}$$

and, in particular, if ϑ be the angle made with $Y=0$ by the tangent to a curve at the point (x, y),

$$e^{2i\vartheta} = \frac{dx}{dy}. \tag{4}$$

We shall use the name *inclination* (or inclination with respect to the axis $Y=0$) for the angle ϑ; and when more than one line is concerned, the quantity $\Sigma\vartheta$* will be spoken of as the *inclination of the system of lines*. The function $e^{2i\vartheta}$ or $e^{2i\Sigma\vartheta}$ may be called the *clinant* (or inclination-function) of the line or system of lines. Thus the clinant of a system of lines is the product of the clinants of the separate lines.

It should be observed that if two sets of n lines each have equal clinants with respect to any axis, they have equal clinants with respect to any axis. To say that two systems of n lines each have equal clinants is the same as to say that they have the same orientation, or that the sum of the angles which the lines of one system make with an arbitrary axis is equal to the like sum for the other system. But relations between clinants other than that of equality—and

* It is understood throughout that $\Sigma\vartheta$ is determined only to modulus π. When two inclinations differ by a multiple of π, they are regarded as equal.

even this relation when the systems do not consist of equal numbers of lines—depend upon the axis of reference.

If the coordinates of a line be defined as the quantities u and v which occur in the equation

$$ux + vy + 1 = 0, \tag{5}$$

it is plain that the inclination of the line is given by the equation

$$e^{2i\theta} = -\frac{v}{u}. \tag{6}$$

The clinant of the system of lines joining the origin to the points (x_1, y_1), $(x_2, y_2), \ldots. (x_n, y_n)$ is

$$e^{2i\Sigma\theta} = \frac{x_1 x_2 \ldots. x_n}{y_1 y_2 \ldots. y_n}, \tag{7}$$

and the clinant of the system of lines $u_1 x + v_1 y + 1 = 0, \ldots., u_n x + v_n y + 1 = 0$ is

$$e^{2i\Sigma\theta} = (-)^n \frac{v_1 v_2 \ldots. v_n}{u_1 u_2 \ldots. u_n}, \tag{8}$$

A third variable will often be introduced for the sake of homogeneity; it will then be understood that $\frac{x}{z}$, $\frac{y}{z}$ must be used for the quantities above denoted by x and y; and $\frac{u}{w}$ and $\frac{v}{w}$ for the quantities denoted by u and v. The points $(1, 0, 0)$, $(0, 1, 0)$ are, of course, the circular points I, J.

II.—The Theorems of Laguerre and Humbert.

The clinant of the asymptotes of the curve

$$A \equiv a_0 x^n + a_1 x^{n-1} y + \ldots. + a_{n-1} x y^{n-1} + a_n y^n + z A_1 = 0$$

is, by (7), $(-)^n \frac{a_n}{a_0}$. In order, then, that the asymptotes of another curve

$$B \equiv b_0 x^n + b_1 x^{n-1} y + \ldots. + b_{n-1} x y^{n-1} + b_n y^n + z B_1 = 0$$

have the same orientation, it is necessary and sufficient that

$$b_n : b_0 = a_n : a_0.$$

But this is also the necessary and sufficient condition that in the pencil

$$\alpha A + \beta B = 0,$$

there be a curve in whose equation the group of terms free of z contains xy as a factor; i. e. a curve which passes through both the circular points. Hence the theorem:

In order that two curves of the n^{th} order have the same asymptotic orientation, it is necessary and sufficient that in the pencil determined by them there be included a circular curve.

If we consider the more general system

$$\alpha A + \beta B + \gamma C + \ldots . + \lambda L = 0,$$

where

$$C \equiv c_0 x^n + \ldots . + c_n y^n + zC_1, \ldots ., L \equiv l_0 x^n + \ldots . + l_n y^n + zL_1,$$

and α, β, ... λ are arbitrary parameters, the necessary and sufficient condition that the asymptotic orientation of the curves of this family be constant is obviously

$$\frac{a_n}{a_0} = \frac{b_n}{b_0} = \ldots . = \frac{l_n}{l_0}.$$

But this is also the necessary and sufficient condition that any curve of the family which passes through I pass also through J. Hence the theorem:*

Given a family of curves of the n^{th} order whose equation involves any number of variable parameters; in order that the orientation of the asymptotes be the same for all these curves, it is necessary and sufficient that every curve of the family which passes through one of the circular points pass also through the other.

To find the clinant of the system of lines joining the origin to the intersections of two curves, we eliminate z between the two equations; the required clinant is the ratio of the coefficient of the highest power of y to the coefficient of the highest power of x in the resulting equation. Now these coefficients are unaffected by the terms containing xy in the equations of the given curves; hence the theorem:

In determining the orientation of the system of lines joining the origin to the intersections of a curve F_n with any other curve, F_n may be replaced by $F_n + xyF_{n-2}$, F_{n-2} being an arbitrary function of x, y, z of the degree $n-2$. This may be otherwise stated as follows:

The orientation of the system of lines joining a given point O to the intersections of a curve F_n with any other curve, is unaltered if F_n be replaced by any curve of

*Given by Humbert, this Journal, X, 260. Humbert, however, takes α, β, ... λ rational functions of one parameter; it is evident that this restriction would not interfere with the above proof.

the n^{th} order through the intersections of F_n with the circular rays of O; and, in particular, if F_n be replaced by *any system of n lines joining each of the n intersections of F_n with OI to one of the intersections of F_n with OJ.* If the curve is real, there is one and only one such system of n lines that is real.

The following special cases may be noticed:

1°. If O is a focus, and if (with Humbert) we designate as a *directrix* the line joining the points of contact of OI and OJ with the curve, the directrix counts twice among the system of n lines. In the case of a conic, therefore, the directrix counting twice entirely replaces the conic; whence the theorem that the inclination of the system of lines joining the focus of a conic to the intersections of the conic with any curve is twice the inclination of the system of lines joining the focus to the intersections of the directrix with the curve. If the cutting curve is a straight line, this becomes a familiar theorem; if the cutting curve is another conic having O for a focus, we see that the inclination of the system of lines joining O to the four intersections of the conics is four times the inclination of the line joining O to the intersection of the directrices.

2°. If O is on the curve, the tangent at O is one of the n replacing lines; if O is a k-ple point on the curve, the k tangents at O constitute k of the n replacing lines.

3°. If the curve passes l times through the circular points, the line at infinity counts l times among the n lines.

It is obvious from 2° and 3° that if we take as axis a line with respect to which the inclination of the tangents at O is 0,* the inclination of the lines joining O to the intersections of the curve F_n with any other curve C is equal to the inclination of the asymptotes of C counted l times, plus the inclination of the lines joining O to the intersection of C with a certain system of $n-k-l$ lines. In particular, if $k+l=n$, the inclination of the system $O(F_n, C)$ is simply l times that of the asymptotes of C. The following particular instances of this last case may be mentioned:

a). The sum of the angles made with a tangent to a circle by the lines joining its point of contact to the intersections of the circle with any curve is equal to the sum of the angles made with the tangent by the asymptotes of the curve.

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* Viz. the tangent itself if O is an ordinary point, a bisector of the angle between the tangents if O is a double point, etc.

b). Given a $\begin{cases}\text{circular cubic}\\ \text{bicircular quartic}\end{cases}$ with a double point O, let OX bisect the angle between the tangents at O. Then the sum of the angles made with OX by the lines joining O to the intersections of the $\begin{cases}\text{cubic}\\ \text{quartic}\end{cases}$ with any curve C is equal to $\begin{cases}\text{the sum}\\ \text{twice the sum}\end{cases}$ of the angles made with OX by the asymptotes of the curve C. Thus, for example, the orientation of the lines joining the double point of a lemniscate to the four intersections of the lemniscate with any transversal depends only on the direction of the transversal; the sum of the angles made by the four lines with an axis of the lemniscate being twice the angle made with the axis by the transversal.

If we make no reference to any particular axis, the theorem of which the foregoing cases are illustrations has the form: If O be a k-ple point on a curve F_n which passes $n-k$ times through the circular points, the orientation of the system of lines joining O to the intersections of F_n with any curve C, depends only on the orientation of the asymptotes of C.

To find the clinant of the system of common tangents of two curves given in tangential coordinates, we should eliminate w between the equations of the curves; the clinant is the ratio of the coefficient of the highest power of u to that of the highest power of v in the resulting equation. But these coefficients are unaffected by the terms containing uv in the equations of the curves; hence, *in determining the orientation of the system of tangents common to a curve F_n and any other curve, F_n may be replaced by $F_n + uvF_{n-2}$*, F_{n-2} being an arbitrary function of u, v, w of the degree $n-2$.

On the other hand, to find a set of foci of a curve F_n not touching the line at infinity, we have to intersect each of the n lines given by the pair of equations

$$u = 0, \quad F_n = 0,$$

with one of the n lines given by the pair of equations

$$v = 0, \quad F_n = 0;$$

whence it is plain that if two curves have the same foci, their equations differ only in the terms involving uv; we have, therefore, this theorem of Laguerre's:

The orientation of the system of common tangents of two curves depends only on the position of the foci, and is therefore the same as that of the system of lines

joining a set of foci of the one to a set of foci of the other. Obviously, this theorem may be regarded as dualistic to that of page 164 (end).

When one (or both) of the curves touches the line at infinity, this theorem requires some modification. If a curve touch the line at infinity at g points $u = \alpha_1 v, \ldots . u = \alpha_g v$, *which do not include I or J*, its equation is of the form

$$(u - \alpha_1 v) \ldots . (u - \alpha_g v)\, w^{n-g} + \text{terms of lower degree in } w = 0;$$

and it is plain that the equation of any other curve having the same finite foci and the same contacts with the line at infinity will differ from this equation only in terms involving uv. Now, if we take any second curve $\Phi_{n'}$, touching the line at infinity at g' points,* the clinant of the system of $nn' - gg'$ common tangents of F_n and $\Phi_{n'}$ *other than the line at infinity* will be obtained as before through the elimination of w between the two equations, and will be unaffected by the terms containing uv in these equations. Hence *the orientation of the system of common tangents of two curves, exclusive of the line at infinity, is unaffected by replacing each curve by any other curve having the same finite foci and the same contacts with the line at infinity*; and, in particular, is the same as that of the system of lines (other than the line at infinity) joining the finite foci and the contacts with the line at infinity of the one curve with the finite foci and the contacts with the line at infinity of the other curve. For example, the three common tangents of two parabolas have the same orientation as the axes of the parabolas and the line joining their foci; and the four common tangents of a parabola and a circle have the same orientation as the axis of the parabola counted twice and the line joining the centre of the circle with the focus of the parabola, counted twice.

If, among the g contacts with the line at infinity the circular points are included, Laguerre's theorem requires further modification; we shall consider, however, only the extreme case of curves whose foci are all at infinity; the general equation of such curves is evidently

$$uvwF_{n-3}(u, v, w) + au^n + \ldots . + bv^n = 0.$$

Now the result of eliminating w between a given equation of this form and the equation of any curve of the class r not touching the line at infinity is evidently

$$a^r u^{nr} + \ldots . + b^r v^{nr} = 0,$$

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* These points I suppose to be different from the g points of contact belonging to F; if any of the latter coincide with any of the former set, complications are introduced which it does not seem worth while to discuss.

so that *the clinant of the system of common tangents, which is* a^r/b^r, *depends only on the value of* r, and is otherwise independent of the nature of the other curve. That is,

Given a curve F *all of whose foci are at infinity, the orientation of the system of tangents common to it and any curve* C *not touching the line at infinity depends only on the class of* C;* so that the orientation of the system of tangents to F through a point is independent of the position of the point, and the orientation of the system of tangents common to F and any curve of the r^{th} class not touching the line at infinity is the same as that of the system of tangets to F through r arbitrary points.

It is worth while to examine more explicitly the case of the tangents drawn *from a point* to a curve.

If the equation of a curve in tangential coördinates is

$$au^n + bv^n + cw^n + \ldots\ldots = 0,$$

the system of tangents through the origin is given by

$$w = 0, \quad au^n + \ldots\ldots + bv^n = 0,$$

and the clinant of this system of lines is a/b.

Observing that a change of origin is effected by replacing w by $w + \alpha u + \beta v$, we see at once that in order that the clinant of the system of tangents drawn to the curve from any point be constant, it is *necessary and sufficient* that the terms in w all contain the product uv; in other words, that the foci of the curve be all at infinity. This is a proof of Humbert's theorem independent of the proof given above.

The systems of tangents through I and J are given by the equations

$$v = 0, \quad au^n + \ldots\ldots + cw^n = 0; \quad u = 0, \quad bv^n + \ldots\ldots + cw^n = 0;$$

*See Humbert, this Journal, X, 263. Humbert gives only the case where the second curve is a point; the more general theorem is, however, an obvious corollary from Humbert's. It should be added that Humbert makes an oversight in saying that curves which have all their foci at infinity touch the line at infinity $n-1$ times and pass through I and J; it is only necessary that they touch the line at infinity at I and J, and that the line at infinity count as *n tangents from* I (and likewise from J). Thus the line at infinity need not be more than a double tangent; and the curve may therefore have $n-2$ tangents parallel to a given line; while if the line at infinity were an $(n-1)$-fold tangent, the curve could have but one tangent parallel to a given line.

so that if $x_1, x_2, \ldots . x_n$ be the x's and $y_1, y_2, \ldots . y_n$ the y's of the foci, we have

$$x_1 x_2 \ldots . x_n = (-)^n \frac{a}{c}, \quad y_1 y_2 \ldots . y_n = (-)^n \frac{b}{c},$$

$$\frac{x_1 x_2 \ldots . x_n}{y_1 y_2 \ldots . y_n} = \frac{a}{b};$$

i. e. the clinant of the system of lines joining the origin to a group of foci is a/b, which is the same as that of the system of tangents from the origin; hence the system of tangents drawn from any point to a curve and the system of lines joining that point to a group of foci, have the same orientation: a particular case of Laguerre's theorem (page 166, end).

Denoting the lengths of the focal radii of the origin—i. e. the lines joining the origin to a set of foci—by $r_1, r_2, \ldots . r_n$, we have, since $r_k^2 = x_k y_k$,

$$(r_1 r_2 \ldots . r_n)^2 = x_1 x_2 \ldots . x_n . y_1 y_2 \ldots . y_n = \frac{ab}{c^2}.$$

If, then, two curves have the coefficients a, b in the same ratio, the orientation of the system of lines joining the origin to a group of foci is the same for the two curves; and if they have the coefficients a, b, c in the same ratio, the product of the focal radii of the origin is also the same for both curves. But the equality of the ratio of a to b for two curves is evidently the condition that in the tangential pencil determined by them there be one which has the origin for a focus; and the equality of the ratios of a, b, c for two curves is the condition that in the tangential pencil determined by them there be one which touches the line at infinity and has the origin for a focus. Hence we have the theorem:

If, from a given focus of a curve belonging to a tangential pencil, lines be drawn to the foci of any curve of the pencil, the orientation of this system of lines is constant; and if the curve to which the given focus belongs touches the line at infinity, the product of the lengths of the lines drawn from it is also constant.* For example, in a system of conics touching four lines, the bisector of the angle formed by the lines joining a focus of a given conic of the system to the two foci of any other conic of the system is a fixed line; and if the given conic is the parabola that belongs to the system, the product of the distances from its focus to the two foci of any other conic of the system is also constant.

*I. e., any complete set of foci, for instance the real foci.

And it is evident, conversely, that

If a point O be such that the systems of lines joining it to the foci of two curves have the same orientation, O is a focus of a curve belonging to the tangential pencil determined by the two curves; and if the product of the lengths of the lines be also the same for the two systems, the curve of which O is a focus touches the line at infinity.

The property contained in the first clause of this and the preceding theorem was given by Humbert (this Journal, X, 262); he points out that it defines by a simple geometrical character the locus of the foci of a tangential pencil.

The equation of this locus is very easily obtained. In fact, the foci of the curve $F(u, v, w) = 0$ are evidently given by the equations $F(z, 0, -x)$, $F(z, 0, -y) = 0$. Hence any focus of a curve belonging to the pencil

$$F(u, v, w) + \lambda\Phi(u, v, w) = 0$$

satisfies the equations

$$F(z, 0, -x) + \lambda\Phi(z, 0, -x) = 0, \quad F(0, z, -y) + \lambda\Phi(0, z, -y) = 0;$$

hence the locus of the foci is

$$F(z, 0, -x)\,\Phi(0, z, -y) - F(0, z, -y)\,\Phi(z, 0, -x) = 0.$$

It is evident that this equation of the $(2n)^{\text{th}}$ degree contains the factor z, and that after striking out this factor, every term is of at least the $(n-1)^{\text{th}}$ degree in x and z and of at least the $(n-1)^{\text{th}}$ degree in y and z.

The tangents to this locus at the circular points are given by the aggregate of terms of lowest degree in (x, z) and (y, z) respectively, that is, by the equations

$$\frac{1}{z}\{F(z, 0, -x)\,\Phi(0, 0, -1) - \Phi(z, 0, -x)\,F(0, 0, -1)\} = 0,$$

$$\frac{1}{z}\{F(0, z, -y)\,\Phi(0, 0, -1) - \Phi(0, z, -y)\,F(0, 0, -1)\} = 0.$$

The intersections of these two pencils of lines are the singular foci of the locus; and these intersections obviously satisfy the equation

$$F(z, 0, -x)\,\Phi(0, z, -y) - F(0, z, -y)\,\Phi(z, 0, -x) = 0,$$

that is, they lie on the curve itself. Hence

The locus of the foci of the curves belonging to the tangential pencil determined by two curves of the n^{th} *class is* (apart from the line at infinity) *a curve of the*

$(2n-1)^{th}$ *order which has the circular points for* $(n-1)$*-ple points and which passes through its own singular foci.**

If the curves F and Φ both touch the line at infinity, or if one of them touch it more than once, the locus degenerates. For example, if the curves F and Φ each have the line at infinity as a simple tangent,

$$F(z,\ 0,\ -x),\quad F(0,\ z,\ -y),\quad \Phi(z,\ 0,\ -x),\quad \Phi(0,\ z,\ -y)$$

each contain z once as a factor; and the locus

$$F(z,\ 0,\ -x)\,\Phi(0,\ z,\ -y) - F(0,\ z,\ -y)\,\Phi(z,\ 0,\ -x) = 0$$

includes the line at infinity counted twice, and *a curve of the* $(2n-2)^{th}$ *order which has the circular points for* $(n-1)$*-ple points.* Thus the locus of the foci of the parabolas which touch three given lines is a circle, and the locus of the foci of the curves of the third class which touch the line at infinity and seven other given lines is a bicircular quartic.

Given a number of points $(x_1,\ y_1),\ \ldots.\ (x_n,\ y_n)$, let the point defined by the equations

$$\frac{1}{x} = \frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots. + \frac{1}{x_n}\right),\quad \frac{1}{y} = \frac{1}{n}\left(\frac{1}{y_1} + \frac{1}{y_2} + \ldots. + \frac{1}{y_n}\right)$$

be called the harmonic centre of the system of points $1,\ 2,\ \ldots.\ n$ with respect to the origin. It is plain that if the points are real (so that x_i and y_i are conjugate imaginaries), the harmonic centre C may be constructed by laying off on $O1,\ O2,\ \ldots.\ On$, distances equal to the reciprocals of $O1,\ O2,\ \ldots.\ On$, and taking for OC the reciprocal of the n^{th} part of the resultant of the system of forces represented by $O1,\ O2,\ \ldots.\ On$.

The direction of the harmonic centre depends only on the ratio of $\sum \frac{1}{x_1}$ to $\sum \frac{1}{y_1}$; its distance only on the product of $\sum \frac{1}{x_1}$ by $\sum \frac{1}{y_1}$.

The harmonic centre $(x,\ y)$ of a set of foci of the curve

$$au^n + a_1 u^{n-1} w + \ldots. + bv^n + b_1 v^{n-1} w + \ldots. = 0,$$

* Humbert (l. c., p. 277) gives this theorem for the particular case of a tangential pencil of conics; he deduces it from a construction of the locus by points, which shows the locus to be a circular cubic on which I and J are corresponding points.

is given by

$$\frac{n}{x} = \sum \frac{1}{x_1} = \frac{a_1}{a}, \quad \frac{n}{y} = \sum \frac{1}{y_1} = \frac{b_1}{b};$$

and for any curve of the tangential pencil determined by the above curve and the curve

$$\alpha u^n + \alpha_1 u^{n-1} w + \ldots + \beta v^n + \beta_1 v^{n-1} w + \ldots = 0$$

the harmonic centre of the foci is given by

$$\frac{n}{x} = \frac{a_1 + \lambda \alpha_1}{a + \lambda \alpha}, \quad \frac{n}{y} = \frac{b_1 + \lambda \beta_1}{b + \lambda \beta},$$

whence, eliminating λ, the locus of the harmonic centres of foci for the curves of the pencil is

$$(a_1\beta_1 - \alpha_1 b_1)\, xy - n\,(a_1\beta - \alpha_1 b)\, x - n\,(a\beta_1 - \alpha b_1)\, y + n^2\,(a\beta - \alpha b) = 0,$$

a circle. Hence the theorem (given by Humbert without demonstration, l. c., p. 281):

The harmonic centre, with respect to a point of the plane, of the real foci of each of the curves of a tangential pencil, describes a circle.

Concerning this locus of harmonic centres, the following points are obvious:

1°. The locus passes through the origin if $a:b = \alpha:\beta$; i. e. if the origin is a focus of a curve of the pencil.

2°. The locus becomes a straight line if $a_1:b_1 = \alpha_1:\beta_1$; i. e. if for one of the curves the harmonic centre is at an infinite distance from the origin.

3°. The harmonic centre is a fixed point for all the curves of the pencil if $a:a_1:b:b_1 = \alpha:\alpha_1:\beta:\beta_1$; i. e. if the origin is a double or multiple focus of some curve of the pencil. Finally, observing that the equation of the locus of the foci of the curves of the pencil (see p. 170) is, apart from the line at infinity,

$$(a\beta - \alpha b)\, z^{2n-1} - \{(a_1\beta - \alpha_1 b)\, x + (a\beta_1 - \alpha b_1)\, y\}\, z^{2n-2} + \ldots = 0,$$

we see that when $a:b = \alpha:\beta$ this curve not only passes through the origin but it has at the origin the same tangent as the circle of harmonic centres; whence this theorem of Humbert's (l. c., p. 276):

Let F be the locus of the foci of the curves of class n belonging to a given tangential pencil: the harmonic centre of the n real foci of any one of these curves with respect to a point arbitrarily chosen upon F remains on a circle touching the curve F at this point.

III.—ON THE DIFFERENTIAL EQUATION

$$\frac{dx}{(x-a_1)^{q_1}(x-a_2)^{q_2}\ldots.(x-a_n)^{q_n}}=\frac{dy}{(y-b_1)^{q_1}(y-b_2)^{q_2}\ldots.(y-b_n)^{q_n}}.$$

General Remarks and Miscellaneous Examples.

The differential equation

$$\frac{dx}{\phi(x)}=\frac{dy}{\psi(y)}, \tag{1}$$

where

$$\phi(x)=(x-a_1)^{q_1}(x-a_2)^{q_2}\ldots(x-a_n)^{q_n},\quad \psi(y)=(y-b_1)^{q_1}(y-b_2)^{q_2}\ldots(y-b_n)^{q_n},$$

admits of a simple and interesting geometrical interpretation if x, y are regarded as circular coordinates. Namely, writing the equation

$$\frac{dx}{dy}=\frac{\phi(x)}{\psi(y)},$$

we see that it defines a curve having the property

$$\vartheta_T=q_1\vartheta_1+q_2\vartheta_2+\ldots.q_n\vartheta_n, \tag{2}$$

ϑ_T being the angle made with the axis of X by the tangent at any point P, and ϑ_k the angle made with the axis of X by the line joining P to the point (a_k, b_k).

Obviously, the curves belonging to the equation

$$\frac{dx}{\phi(x)}+\frac{dy}{\psi(y)}=0 \tag{3}$$

have the property (ϑ_N denoting the inclination of the normal)

$$\vartheta_N=q_1\vartheta_1+q_2\vartheta_2+\ldots.+q_n\vartheta_n, \tag{4}$$

and are the orthogonal trajectories of the former set of curves; and, more generally, the oblique trajectories of the curves

$$\frac{dx}{\phi(x)}=\frac{dy}{\psi(y)},$$

the angle of intersection being α, are given by

$$e^{-i\alpha}\frac{dx}{\phi(x)}=e^{i\alpha}\frac{dy}{\psi(y)}. \tag{5}$$

The curves defined by the last equation have the property

$$\vartheta_T=\alpha+q_1\vartheta_1+q_2\vartheta_2+\ldots.+q_n\vartheta_n. \tag{6}$$

When the q's are all integers, positive or negative, ϕ and ψ are rational and the equation is integrable; hence we can always obtain the equation of a curve defined by the property that the angle made by its tangent with a fixed line is a sum of integer multiples of the angles made with that line by the rays from the point of contact to n fixed points, which we may call direction-centres. If the q's, besides being integers, are such that

$$q_1 + q_2 + \ldots . + q_n = 1,$$

the equation giving the inclination of the tangent will not be affected by changing the directions of the axes of reference; and the property of the tangent of the curve given by equation (1) may be stated thus: the system of lines consisting of the tangent and the rays from its point of contact to the direction-centres corresponding to negative exponents, has the same orientation as the system of rays from the point of contact to the direction-centres corresponding to positive exponents; each direction-centre being counted a number of times equal to the absolute value of the corresponding exponent.

Let us consider some examples.

1°. $\dfrac{dx}{x} - \dfrac{dy}{y} = 0$. This equation defines a curve in which the *tangent coincides with the radius vector.* Its solution is $y = cx$ or $Y = CX$, a straight line through the origin.

2°. $\dfrac{dx}{x} + \dfrac{dy}{y} = 0$; orthogonal trajectory of preceding; *normal coincides with radius vector.* Solution: $xy = c$ or $X^2 + Y^2 = C$, a circle with its centre at the origin.

3°. $\dfrac{xdx}{x^2 - a^2} - \dfrac{ydy}{y^2 - a^2} = 0$; *bisector of angle formed by tangent and radius vector coincides with bisector of angle formed by rays to the two points* $X = \pm a$, $Y = 0$. Solution: $x^2 - a^2 = c(y^2 - a^2)$, or $X^2 - Y^2 - a^2 = CXY$, an equilateral hyperbola passing through the two points and having its centre at the origin.

4°. $\dfrac{xdx}{x^2 - a^2} + \dfrac{ydy}{y^2 - a^2} = 0$; orthogonal trajectory of preceding; *bisector of angle formed by normal and radius vector coincides with bisector of angle formed by rays to the two points* $X = \pm a$, $Y = 0$. Solution: $(x^2 - a^2)(y^2 - a^2) = c$, or $(X^2 + Y^2)^2 - 2a^2(X^2 - Y^2) = C$, a Cassinian with the two points for foci.

5°. $\frac{x^{n-1}dx}{x^n - a^n} - \frac{y^{n-1}dy}{y^n - a^n} = 0$; *orientation of tangent and* $(n-1)$ *times radius vector is equal to orientation of rays to a system of n points (beginning with the point* $X = a$, $Y = 0$) *uniformly distributed on a circle whose centre is the origin.* Solution: $x^n - a^n = c(y^n - a^n)$; or, in polar coordinates, $r^n \cos n\vartheta - a^n = Cr^n \sin n\vartheta$.

6°. $\frac{x^{n-1}dx}{x^n - a^n} + \frac{y^{n-1}dy}{y^n - a^n} = 0$; orthogonal trajectory of preceding; *orientation of normal and* $(n-1)$ *times radius vector is equal to orientation of rays to the system of n points defined in* 5°. Solution: $(x^n - a^n)(y^n - a^n) = c$; or, in polar coordinates, $r^{2n} - 2a^n r^n \cos n\vartheta = C$. The curves $r^n = 2a^n \cos n\vartheta$, $r^n \cos n\vartheta = a^n$, are particular solutions of 6° and 5° respectively; when $n = 2$, these are a lemniscate and an equilateral hyperbola with its vertices at the foci of the lemniscate.

Of course, whenever $q_1 + q_2 + \ldots + q_n = 1$, whether the q's be integers or not, the property of the tangent is independent of the direction of reference; e. g., the equation

$$\frac{dx}{\sqrt{a + 2bx + cx^2}} + \frac{dy}{\sqrt{a + 2by + cy^2}} = 0$$

gives a curve whose normal bisects the angle between the rays drawn to two fixed points; and, more generally, the equation

$$\frac{x^{n-1}dx}{\sqrt{P_{2n}(x)}} + \frac{y^{n-1}dy}{\sqrt{P_{2n}(y)}} = 0,$$

$P_{2n}(x)$ being a polynomial of the $(2n)^{\text{th}}$ degree in x, gives a curve such that the sum of the angles made with *any* line by the normal and by the radius vector counted $(n-1)$ times is half the sum of the angles made with that line by the rays to $2n$ fixed points.

If the sum of the q's is not equal to 1, the equation connecting the inclination of the tangent with the inclinations of the rays to the direction-centres will be modified if the direction of reference is altered; the modification consists, however, merely in adding a constant to the value of ϑ_T.

In the examples that follow, when the word *inclination* is used, it is understood to have reference to the axis of X.

7°. $\frac{dx}{x^{n+1}} - \frac{dy}{y^{n+1}} = 0$; *inclination of tangent* $= (n+1)$ *times inclination of radius vector.* Solution: $x^{-n} - y^{-n} = c$, or $r^n = C^n \sin n\vartheta$.

8°. $\frac{dx}{x^{n+1}} + \frac{dy}{y^{n+1}} = 0$;* *inclination of normal* $= n + 1$ *times inclination of radius vector.* Solution: $r^n = C^n \cos n\vartheta$. Of course, in 7° and 8°, n need not be an integer.

We may remark that a particular solution of 8° coincides with a particular solution of 6°, viz. the solution $r^n = 2a^n \cos n\vartheta$; in this curve, then, we have, if n is a positive integer,

$$\vartheta_N = (n+1)\,\vartheta_0 \text{ and } \vartheta_N + (n-1)\,\vartheta_0 = \vartheta_1 + \vartheta_2 + \ldots + \vartheta_n,$$

whence also
$$2n\vartheta_0 = \vartheta_1 + \vartheta_2 + \ldots + \vartheta_n;$$

so that, for instance, in the lemniscate the sum of the angles made by the focal radii with the axis is 4 times the angle made by the radius vector with the axis.

We may here notice a point concerning the Cassinian generally. The equation being written

$$(X^2 + Y^2)^2 - 2a^2(X^2 - Y^2) = b^4 - a^4,$$

the two foci, 1 and 2, by means of which the curve is usually defined, are on the axis of X at the distance a on either side of the origin; and it is evident on inspection of the equation written in the form

$$x^2y^2 - a^2(x^2 + y^2)z^2 + (a^4 - b^4)z^4 = 0$$

that these two foci are at the same time the two singular foci† and two of the ordinary foci, the remaining two ordinary foci, 3 and 4, being on the axis of Y at the distance $\sqrt{b^4 - a^4}/a$ on either side of the origin if $b > a$, and on the axis of X at the distance $\sqrt{a^4 - b^4}/a$ on either side of the origin if $b < a$; of course, if $b = a$, i. e. if the Cassinian is a lemniscate, 3 and 4 coincide with the origin. Now by a known property belonging to bicircular quartics in general,

$$2\vartheta_N = \vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4,$$

and by example 6°

$$\vartheta_N + \vartheta_0 = \vartheta_1 + \vartheta_2.$$

Hence
$$\vartheta_N - \vartheta_0 = \vartheta_3 + \vartheta_4,$$

and
$$2\vartheta_0 = \vartheta_1 + \vartheta_2 - \vartheta_3 - \vartheta_4.$$

* This is, of course, the orthogonal trajectory of the preceding; but it is also plain that the curves corresponding to this equation are simply those corresponding to 7° turned through an angle $\frac{\pi}{2n}$.

† I. e., foci obtained by intersecting tangents *at* I and J. It is because I and J are inflexions that these foci play the double part in the Cassinian.

In the particular case of the lemniscate, $\vartheta_3 = \vartheta_4 = \vartheta_0$, and this equation becomes $4\vartheta_0 = \vartheta_1 + \vartheta_2$, as otherwise found above.

If $\int \frac{dx}{f(x)} = F^{-1}(x)$, and if $F^{-1}(X - iY)$ may be taken the conjugate of $F^{-1}(X + iY)$, then writing

$$F^{-1}(X + iY) = u + iv,$$

the solution of
$$e^{-i\alpha}\frac{dx}{f(x)} + e^{i\alpha}\frac{dy}{f(y)} = 0$$

may be written
$$u\cos\alpha + v\sin\alpha = C.$$

If, further, we write

$$F(u + iv) = F_1(u, v) + iF_2(u, v),$$

the curve is given in rectangular coordinates by the equations

$$X = F_1(u, v), \quad Y = F_2(u, v),$$

the parameters being connected by the relation

$$u\cos\alpha + v\sin\alpha = C.$$

The next two examples may serve as illustrations of the foregoing.

9°. $e^{-i\alpha}\frac{dx}{\sqrt{k^2 - x^2}} + e^{i\alpha}\frac{dx}{\sqrt{k^2 - y^2}} = 0$; *normal makes an angle α with bisector of rays to points $X = \pm k$, $Y = 0$.* Here we may take

$$F(u + iv) = k\cos(u + iv) = k(\cos u \cosh v - i\sin u \sinh v);$$

hence the curves are given by

$$\begin{cases} X = k\cos u\cosh v, \quad Y = -k\sin u\sinh v, \\ \qquad u\cos\alpha + v\sin\alpha = C. \end{cases}$$

If $\alpha = 0$, $u = C$; if $\alpha = \frac{\pi}{2}$, $v = C$; in these cases the equation of the curves is evidently

$$\frac{X^2}{(k\cos C)^2} - \frac{Y^2}{(k\sin C)^2} = 1, \quad \frac{X^2}{(k\cosh C)^2} + \frac{Y^2}{(k\sinh C)^2} = 1,$$

either of which represents a system of confocal ellipses and hyperbolas; either equation is converted into the other by the substitution of iC for C.

It is easy to eliminate u and v in the general case; viz. we have

$$2\frac{X^2+Y^2}{k^2}=2(\cos^2 u\cosh^2 v+\sin^2 u\sinh^2 v)=\cos 2u+\cosh 2v,$$

$$2\frac{X^2-Y^2}{k^2}=2(\cos^2 u\cosh^2 v-\sin^2 u\sinh^2 v)=1+\cos 2u\cosh 2v,$$

so that $\cos 2u$, $\cosh 2v$ are the roots of the equation

$$k^2\lambda^2-2(X^2+Y^2)\lambda+2(X^2-Y^2)-k^2=0.$$

The equation of the curve is therefore

$$\cos\alpha\cos^{-1}\lambda_1+\sin\alpha\cosh^{-1}\lambda_2=C,$$

λ_1 and λ_2 being the roots of the preceding equation; or, more explicitly,

$$\cos\alpha\cos^{-1}\frac{X^2+Y^2-\sqrt{(X^2+Y^2)^2-2k^2(X^2-Y^2)+k^4}}{k^2}$$
$$+\sin\alpha\cosh^{-1}\frac{X^2+Y^2+\sqrt{(X^2+Y^2)^2-2k^2(X^2-Y^2)+k^4}}{k^2}=C.$$

When $\alpha=0$ or $\frac{\pi}{2}$, one value of λ is constant, and the equation of the curves may be obtained by putting $\lambda=c$ in the quadratic equation for λ; when we get

$$k^2c^2-2(X^2+Y^2)c+2(X^2-Y^2)-k^2=0,$$
$$2X^2(c-1)+2Y^2(c+1)=k^2(c^2-1),$$

or

$$\frac{X^2}{\frac{1}{2}k^2(c+1)}+\frac{Y^2}{\frac{1}{2}k^2(c-1)}=1,$$

a system of conics with their foci at $X=\pm k$, $Y=0$.

10°. $e^{-i\alpha}\dfrac{dx}{x^2-k^2}+e^{i\alpha}\dfrac{dy}{y^2-k^2}=0$; *normal makes an angle α with line whose inclination = inclination of the pair of rays to the points $X=\pm k$, $Y=0$.* Here we may take

$$F(u+iv)=k\tanh(u+iv)=k\frac{\sinh u\cos v+i\cosh u\sin v}{\cosh u\cos v+i\sinh u\sin v}$$
$$=k\frac{\sinh u\cosh u+i\sin v\cos v}{\cosh^2 u\cos^2 v+\sinh^2 u\sin^2 v}$$
$$=k\frac{\sinh 2u+i\sin 2v}{\cosh 2u+\cos 2v}.$$

Hence the curves are given by

$$\begin{cases} X = k\dfrac{\sinh 2u}{\cosh 2u + \cos 2v}, \quad Y = k\dfrac{\sin 2v}{\cosh 2u + \cos 2v}, \\ u\cos\alpha + v\sin\alpha = C. \end{cases}$$

The parameters u and v are easily eliminated as follows:

$$\frac{X^2+Y^2}{k^2} = \frac{\sinh^2 2u + \sin^2 2v}{(\cosh 2u + \cos 2v)^2} = \frac{\cosh^2 2u - \cos^2 2v}{(\cosh 2u + \cos 2v)^2} = \frac{\cosh 2u - \cos 2v}{\cosh 2u + \cos 2v}.$$

Hence

$$\frac{\cosh 2u}{\cosh 2u + \cos 2v} = \frac{X^2+Y^2+k^2}{2k^2}, \quad \frac{\cos 2v}{\cosh 2u + \cos 2v} = \frac{X^2+Y^2-k^2}{-2k^2},$$

so that, since the original equations give

$$\frac{\sinh 2u}{\cosh 2u + \cos 2v} = \frac{X}{k}, \quad \frac{\sin 2v}{\cosh 2u + \cos v} = \frac{Y}{k},$$

we have
$$\tanh 2u = \frac{2kX}{X^2+Y^2+k^2}, \quad \tan 2v = \frac{-2kY}{X^2+Y^2-k^2};$$

substituting these values in the equation $u\cos\alpha + v\sin\alpha = C$, we have the equation of the curves:

$$\cos\alpha \tanh^{-1}\frac{2kX}{X^2+Y^2+k^2} - \sin\alpha\tan^{-1}\frac{2kY}{X^2+Y^2-k^2} = C.$$

If $\alpha = \frac{\pi}{2}$, this becomes

$$X^2+Y^2-k^2 = cY,$$

a system of circles through the points $Y=0$, $X = \pm k$. If $\alpha = \frac{\pi}{2}$, it becomes

$$X^2+Y^2+k^2 = cX,$$

the orthogonal system of circles, viz. those through $X=0$, $Y = \pm ik$, the anti-points of the preceding pair of points.

Thus the oblique trajectory of a system of circles through two fixed points, like the oblique trajectory of a system of confocal ellipses, is a transcendental curve; the equations of the trajectories being, as found above, respectively

$$\cos\alpha\tanh^{-1}\frac{2kX}{X^2+Y^2+k^2} - \sin\alpha\tan^{-1}\frac{2kY}{X^2+Y^2-k^2} = C,$$

$$\cos\alpha\cos^{-1}\frac{X^2+Y^2-\sqrt{(X^2+Y^2)^2-2k^2(X^2-Y^2)+k^4}}{k^2}$$

$$+\sin\alpha\cosh^{-1}\frac{X^2+Y^2+\sqrt{(X^2+Y^2)^2-2k^2(X^2-Y^2)+k^4}}{k^2} = C.$$

But the former reduces to an algebraic curve in the particular case when the two common-points of the circles coincide, while the latter does not do so when the two foci coincide. To get the limiting form of the first equation when $k = 0$, we may write it

$$\cos\alpha \cdot \frac{1}{k}\tanh^{-1}\frac{2kX}{X^2+Y^2+k^2} - \sin\alpha \cdot \frac{1}{k}\tan^{-1}\frac{2kY}{X^2+Y^2-k^2} = C',$$

which, when $k = 0$, becomes

$$\frac{2X\cos\alpha}{X^2+Y^2} - \frac{2Y\sin\alpha}{X^2+Y^2} = C',$$

or

$$C'(X^2+Y^2) = 2(X\cos\alpha - Y\sin\alpha),$$

a system of circles touching the line $X\cos\alpha - Y\sin\alpha = 0$ at the origin.

The second equation may be written, if we put, for brevity,

$$X^2+Y^2 = p, \quad X^2 - Y^2 = q,$$

$$\cos\alpha\cos^{-1}\frac{p-\sqrt{p^2-2k^2q+k^4}}{k^2} + \sin\alpha\left(\cosh^{-1}\frac{p+\sqrt{p^2-2k^2q+k^4}}{k^2} - \cosh^{-1}\frac{2c}{k^2}\right) = 0;$$

when k is infinitesimal, this may be written

$$\cos\alpha\cos^{-1}\frac{q}{p} + \sin\alpha\left(\cosh^{-1}\frac{2p}{k^2} - \cosh^{-1}\frac{2c}{k^2}\right) = 0,$$

$$\cos\alpha\cos^{-1}\frac{q}{p} + \sin\alpha\cosh^{-1}\left(\frac{2p}{k^2}\cdot\frac{2c}{k^2} - \sqrt{\frac{4p^2}{k^4}-1}\sqrt{\frac{4c^2}{k^4}-1}\right) = 0,$$

which gives, in the limit,

$$\cos\alpha\cos^{-1}\frac{q}{p} + \sin\alpha\cosh^{-1}\left(\frac{p}{2c}+\frac{c}{2p}\right) = 0.$$

Now $\cosh^{-1}\left(\frac{p}{2c}+\frac{c}{2p}\right) = \log\frac{p}{c} = \log\frac{X^2+Y^2}{c} = \log\frac{r^2}{c} = 2\log\frac{r}{c'}$; and $\cos^{-1}\frac{q}{p} = \cos^{-1}\frac{X^2-Y^2}{X^2+Y^2} = \cos^{-1}(\cos 2\vartheta) = 2\vartheta$; hence the above equation becomes (in polar coordinates)

$$r = c'e^{-\vartheta\cot\alpha},$$

a system of similar logarithmic spirals. Of course, the trajectories for the case when the two direction-centres coincide are in both instances obvious otherwise.

11°. If a curve be defined by the property that the inclination of the tangent is equal to half the inclination of the system of rays drawn from its point of contact to four arbitrary fixed points, its differential equation is

$$\frac{dx}{\sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}}=\frac{dy}{\sqrt{(y-b_1)(y-b_2)(y-b_3)(y-b_4)}},$$

the four points being $(a_1, b_1) \ldots . (a_4, b_4)$.

Denoting the function inverse to the integral of the first member by ϕ, and the function inverse to the integral of the second member by ψ, the solution of this equation is

$$\phi^{-1}(x)-\psi^{-1}(y)=C.$$

If the four points are real, $b_1 \ldots . b_4$ are the conjugates of $a_1 \ldots . a_4$ respectively; and consequently if we write $\phi^{-1}(X+iY)=u_1+iu_2$, we may put $\psi^{-1}(X-iY)=u_1-iu_2$, so that the foregoing equation becomes

$$u_2=c.$$

Writing, then,

$$\phi(u_1+iu_2)=\phi_1(u_1, u_2)+i\phi_2(u_1, u_2),$$

the curve is given in rectangular coordinates by the equations

$$\begin{cases} X=\phi_1(u_1, c), \\ Y=\phi_2(u_1, c). \end{cases}$$

The solution of the problem thus depends solely on the separation of $\phi(u_1+iu_2)$ into its real and pure imaginary components: ϕ being the function inverse to the general elliptic integral of the first kind, $\int dx/\sqrt{R(x)}$, where $R(x)$ is a quartic whose coefficients are in general complex.

It would have been no more difficult to consider the oblique trajectories of these curves; but in point of fact these furnish nothing new, since by rotation of the axes of reference they are reducible to the case just considered.*

* This is not the same as to say that *with a given set of four points and a given axis of reference*, the oblique trajectories are of the same nature as the original curves: which is not true. But with a proper choice of the axis of reference, any proposed case of the oblique trajectory is reduced to *another* case of the original class of curves.

Nor would it be more difficult to consider the case when the four arbitrary points are not real. The proposed curves would in this case be given by

$$\begin{cases} X = \phi_1(u_1, u_2) = \quad \psi_1(v_1, v_2), \\ Y = \phi_2(u_1, u_2) = -\psi_2(v_1, v_2), \\ \quad u_1 - v_1 + i(u_2 - v_2) = 0, \end{cases}$$

ψ_1 and ψ_2 being of course defined by the equation

$$\psi(v_1 + iv_2) = \psi_1(v_1, v_2) + i\psi_2(v_1, v_2).$$

If the points $(a_1, b_1) \ldots\ldots (a_4, b_4)$ are concyclic and are such that an "axis" of the four points is parallel to the axis of X, it is known* that the required curve is a bicircular quartic of which the four points are a set of ordinary foci. Hence in this case the problem of solving the differential equation may be reduced to that of finding the equation of a system of confocal bicircular quartics whose foci are any four concyclic points. It is evident that the condition that the four given points be concyclic is equivalent to the requirement that the quartic in x be linearly transformable into the quartic in y; so that the case to which this geometrical method of solving the differential equation is applicable is that in which the two quartics are homographic and satisfy a certain additional condition. It is obvious, moreover, that *all* the conditions of the case are expressed by the equations

$$\frac{(a_1 - a_2)(a_3 - a_4)}{(b_1 - b_2)(b_3 - b_4)} = \frac{(a_1 - a_3)(a_2 - a_4)}{(b_1 - b_3)(b_2 - b_4)} = \frac{(a_1 - a_4)(a_2 - a_3)}{(b_1 - b_4)(b_2 - b_3)} = 1,$$

viz. the equation of these fractions to 1 expresses that the inclination of the pair of lines 12, 34 is 0, and likewise for the pairs 13, 24 and 14, 23; and it follows incidentally that the four points are concyclic. This follows either by a well-known theorem of geometry, or from the fact that the equality of any two of the fractions shows the quartics to be homographic and the points consequently to be concyclic.

On the Equation $\sin x\, dx = \sin y\, dy$.

It is somewhat interesting to consider a case in which $f(x)$ is a convergent infinite product, as e. g. in the equation

$$\sin x\, dx = \sin y\, dy.$$

*See "Note on the Double Periodicity of the Elliptic Functions," this Journal, XI, 285.

This equation defines a curve in which the angle made by the tangent with the axis of X is the negative of the sum of the angles made with the axis of X by the rays drawn to the points (on that axis)

$$X = 0,\ \pm \pi,\ \pm 2\pi,\ \pm 3\pi,\ \ldots\ ;$$

or, as we may write it,

$$-\vartheta_T = \vartheta_0 + \vartheta_1 + \vartheta_{-1} + \vartheta_2 + \vartheta_{-2} + \vartheta_3 + \vartheta_{-3} + \ldots ^{*}$$

The curves thus defined have for their equation

$$\cos x - \cos y = c, \text{ i. e. } \sin X \sinh Y = C.$$

If we compare the value of the inclination of the tangent as derived from this last equation with that which arises from the definition of the curve, we obtain an interesting identity, viz. we get in these ways respectively

$$\vartheta_T = \tan^{-1}\frac{dY}{dX} = -\tan^{-1}\frac{\cos X \sinh Y}{\sin X \cosh Y} = -\tan^{-1}\frac{\tanh Y}{\tan X},$$

$$-\vartheta_T = \tan^{-1}\frac{Y}{X} + \tan^{-1}\frac{Y}{X-\pi} + \tan^{-1}\frac{Y}{X+\pi} + \tan^{-1}\frac{Y}{X-2\pi} + \tan^{-1}\frac{Y}{X+2\pi} + \ldots$$

$$= \tan^{-1}\frac{Y}{X} + \tan^{-1}\frac{2XY}{X^2 - Y^2 - \pi^2} + \tan^{-1}\frac{2XY}{X^2 - Y^2 - (2\pi)^2} + \ldots ;$$

whence the identity

$$\tan^{-1}\frac{\tanh Y}{\tan X} \equiv \tan^{-1}\frac{Y}{X} + \tan^{-1}\frac{2XY}{X^2 - Y^2 - \pi^2} + \tan^{-1}\frac{2XY}{X^2 - Y^2 - (2\pi)^2} + \ldots \pmod{\pi}^{\dagger} \quad \text{(A)}$$

or

$$\tanh Y = \tan X \tan\left\{\tan^{-1}\frac{Y}{X} + \tan^{-1}\frac{2XY}{X^2 - Y^2 - \pi^2} + \tan^{-1}\frac{2XY}{X^2 - Y^2 - (2\pi)^2} + \ldots\right\}.$$

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* From the nature of the case (viz. because the equation $dx/dy = \sin y/\sin x$ determines a definite direction of the tangent, and because $\sin y/\sin x$ is truly the limit of $\frac{y}{x}\cdot\frac{y-\pi}{x-\pi}\cdot\frac{y+\pi}{x+\pi}\cdot\frac{y-2\pi}{x-2\pi}\cdot\frac{y+2\pi}{x+2\pi}\ldots$, the factors being taken in this order) the series to the right converges to a definite limit, *modulo* π; and it is plainly best to take each pair of terms, $\vartheta_r + \vartheta_{-r}$, as near 0 as possible; then $\lim\,(\vartheta_n + \vartheta_{-n}) = 0$, obviously.

† It is needless to say that this identity is easily obtained without the intervention of geometry; but it is naturally suggested by the construction of the tangent to the curve we are considering.

In particular, when $X = Y$, we have

$$\tan^{-1}\frac{\tanh X}{\tan X} \equiv \frac{\pi}{4} - \tan^{-1}\frac{2X^2}{\pi^2} - \tan^{-1}\frac{2X^2}{(2\pi)^2} - \tan^{-1}\frac{2X^2}{(3\pi)^2} - \dots \pmod{\pi} \quad \text{(B)}$$

and when $X = \frac{\pi}{2}$,

$$\tan^{-1}\frac{1}{2.1^2} + \tan^{-1}\frac{1}{2.2^2} + \tan^{-1}\frac{1}{2.3^2} + \tan^{-1}\frac{1}{2.4^2} + \dots = \frac{\pi}{4} + \lambda\pi,$$

λ being an undetermined integer. If we take each of the inverse tangents at its least positive value $\left(\text{i. e. between } 0 \text{ and } \frac{1}{2}\pi\right)$, it is easy to see that $\lambda = 0$; for the series on the left is less than $\frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$, i. e. less than $\frac{\pi^2}{12}$, which is itself less than $\frac{\pi}{4} + \pi$.

In like manner, if we put $X = \frac{3}{2}\pi$, we obtain

$$\tan^{-1}\frac{3^2}{2.1^2} + \tan^{-1}\frac{3^2}{2.2^2} + \tan^{-1}\frac{3^2}{2.3^2} + \dots = \frac{\pi}{4} + \lambda\pi;$$

and (with the same understanding concerning $\tan^{-1}$) $\lambda = 1$; for the series on the left is greater than the one above considered, but less than

$$\tan^{-1}\frac{3^2}{2.1^2} + \tan^{-1}\frac{3^2}{2.2^2} + 3\left\{\tan^{-1}\frac{3^2}{2.3^2} + \tan^{-1}\frac{3^2}{2.6^2} + \tan^{-1}\frac{3^2}{2.9^2} + \dots\right\},$$

which is itself less than

$$\frac{\pi}{2} + \frac{\pi}{2} + 3\,\frac{\pi}{4},\ = \frac{3\pi}{4} + \pi,\ < \frac{\pi}{4} + 2\pi.$$

Likewise

$$\tan^{-1}\frac{5^2}{2.1^2} + \tan^{-1}\frac{5^2}{2.2^2} + \tan^{-1}\frac{5^2}{2.3^2} + \dots = \frac{\pi}{4} + \lambda\pi,$$

and $\lambda = 2$ because the series on the left is less than

$$\tan^{-1}\frac{5^2}{2.1^2} + \tan^{-1}\frac{5^2}{2.2^2} + \tan^{-1}\frac{5^2}{2.3^2} + \tan^{-1}\frac{5^2}{2.4^2}$$
$$+ 5\left\{\tan^{-1}\frac{5^2}{2.5^2} + \tan^{-1}\frac{5^2}{2.10^2} + \tan^{-1}\frac{5^2}{2.15^2} + \dots\right\},$$
$$< \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + 5\,\frac{\pi}{4},\ = \frac{\pi}{4} + 3\pi.$$

This suggests that when $X = (2k+1)\dfrac{\pi}{2}$, the value of λ is k, giving the formula

$$\tan^{-1}\frac{(2k+1)^2}{2.1^2} + \tan^{-1}\frac{(2k+1)^2}{2.2^2} + \tan^{-1}\frac{(2k+1)^2}{2.3^2} + \ldots = \frac{\pi}{4} + k\pi,$$

k being any positive integer, and $\tan^{-1}$ being always taken between 0 and $\dfrac{\pi}{2}$. But it is evident that the above method will not furnish this result; when $2k+1 \gtreqless 7$, the superior limit arrived at by it will exceed $\dfrac{\pi}{4} + k\pi$. But the formula would be proved if we could show that

$$\tan^{-1}\frac{n^2}{2r^2} + \tan^{-1}\frac{n^2}{2r'^2} + \tan^{-1}\frac{n^2}{2(n+r)^2} + \tan^{-1}\frac{n^2}{2(n+r')^2}$$
$$+ \tan^{-1}\frac{n^2}{2(2n+r)^2} + \tan^{-1}\frac{n^2}{2(2n+r')^2} + \ldots = \pi,$$

where $n = 2k+1$ and $r + r' = n$. This last formula may be more compactly written

$$\sum_{-\infty}^{+\infty}{}_{\nu} \tan^{-1}\frac{n^2}{2(\nu n + r)^2} = \pi.$$

In point of fact it is easy to prove the more general formula

$$\sum_{-\infty}^{+\infty}{}_{\nu} \tan^{-1}\frac{\alpha^2}{2(\nu\alpha + \beta)^2} = \pi,$$

α and β being any real quantities. There is evidently no loss of generality in supposing, in the demonstration, that α is positive, and that β is positive and less than α. We have seen above that

$$\tan^{-1}\frac{\tanh Y}{\tan X} \equiv \tan^{-1}\frac{Y}{X} + \sum_{1}^{\infty}{}_{\nu}\left(\tan^{-1}\frac{Y}{X - \nu\pi} + \tan^{-1}\frac{Y}{X + \nu\pi}\right) (\mathrm{mod}\ \pi);$$

writing, in this, $\dfrac{X - \beta\pi}{\alpha}$ for X and $\dfrac{Y}{\alpha}$ for Y, it becomes

$$\tan^{-1}\frac{\tanh\dfrac{Y}{\alpha}}{\tan\dfrac{X-\beta\pi}{\alpha}} \equiv \tan^{-1}\frac{Y}{X - \beta\pi}$$
$$+ \sum_{1}^{\infty}{}_{\nu}\left\{\tan^{-1}\frac{Y}{X - (\nu\alpha + \beta)\pi} + \tan^{-1}\frac{Y}{X + (\nu\alpha - \beta)\pi}\right\} (\mathrm{mod}\ \pi).$$

Likewise

$$\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X+\beta\pi}{\alpha}}\equiv\tan^{-1}\frac{Y}{X+\beta\pi}$$

$$+\sum_1^\infty\left\{\tan^{-1}\frac{Y}{X-(\nu\alpha-\beta)\pi}+\tan^{-1}\frac{Y}{X+(\nu\alpha+\beta)\pi}\right\}\ (\mathrm{mod}\ \pi).$$

Hence, by addition,

$$\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X-\beta\pi}{\alpha}}+\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X+\beta\pi}{\alpha}}\equiv\tan^{-1}\frac{Y}{X-\beta\pi}+\tan^{-1}\frac{Y}{X+\beta\pi}$$

$$+\sum_1^\infty\left\{\tan^{-1}\frac{Y}{X-(\nu\alpha+\beta)\pi}+\tan^{-1}\frac{Y}{X+(\nu\alpha+\beta)\pi}\right.$$

$$\left.+\tan^{-1}\frac{Y}{X-(\nu\alpha-\beta)\pi}+\tan^{-1}\frac{Y}{X+(\nu\alpha-\beta)\pi}\right\}\ (\mathrm{mod}\ \pi),$$

and it is plain that this may be written

$$\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X-\beta\pi}{\alpha}}+\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X+\beta\pi}{\alpha}}$$

$$\equiv\sum_{-\infty}^{+\infty}\tan^{-1}\frac{2XY}{X^2-Y^2-(\nu\alpha+\beta)^2\pi^2}\ (\mathrm{mod}\ \pi)\ (\mathrm{C})$$

Putting, in this identity, $X=Y=\alpha\frac{\pi}{2}$, the first member is $0\ (\mathrm{mod}\ \pi)$ because $\tan\left(\frac{1}{2}-\frac{\beta}{\alpha}\right)\pi=-\tan\left(\frac{1}{2}+\frac{\beta}{\alpha}\right)\pi$; hence

$$\sum_{-\infty}^{+\infty}\tan^{-1}\frac{\alpha^3}{2(\nu\alpha+\beta)^2}\equiv 0\ (\mathrm{mod}\ \pi).$$

Finally, to determine the value of the series exactly (each term being taken

between 0 and $\frac{\pi}{2}$) we have only to observe that the sum from $\nu = 1$ to $\nu = +\infty$ can not exceed $\sum_1^\infty \tan^{-1}\frac{1}{2\nu^2}$, which has been proved to be equal to $\frac{1}{4}\pi$; that the sum from $\nu = -2$ to $\nu = -\infty$ is less than $\sum_1^\infty \tan^{-1}\frac{1}{2\nu^2}$; and the sum of the two omitted terms ($\nu = 0$, $\nu = 1$) must be less than twice $\frac{1}{2}\pi$. Hence the proposed series is less than $\frac{1}{4}\pi + \frac{1}{4}\pi + \pi$, and therefore less than 2π. Hence

$$\sum_{-\infty}^{+\infty} \tan^{-1}\frac{\alpha^2}{2(\nu\alpha + \beta)^2} = \pi, \tag{D}$$

α and β being any real quantities; Q. E. D. We have thus completed the proof of the equation

$$\tan^{-1}\frac{(2k+1)^2}{2.1^2} + \tan^{-1}\frac{(2k+1)^2}{2.2^2} + \tan^{-1}\frac{(2k+1)^2}{2.3^2} + \ldots = \frac{\pi}{4} + k\pi. \tag{E}$$

If, in formula B, we put $X = 2k . \frac{\pi}{2}$, we get

$$\tan^{-1}\frac{(2k)^2}{2.1^2} + \tan^{-1}\frac{(2k)^2}{2.2^2} + \tan^{-1}\frac{(2k)^2}{2.3^2} + \ldots \equiv -\frac{\pi}{4} \pmod{\pi},$$

and equation D enables us immediately to determine the actual value of the series; viz., putting $\alpha = 2k$, and β successively $= 0, 1, 2, \ldots, 2k-1$, and adding the results, we get the preceding series doubled, and in addition a term $\frac{\pi}{2}$ (arising from $\beta = 0$, $\nu = 0$); hence

$$\tan^{-1}\frac{(2k)^2}{2.1^2} + \tan^{-1}\frac{(2k)^2}{2.2^2} + \tan^{-1}\frac{(2k)^2}{2.3^2} + \ldots$$
$$= \frac{1}{2}\left(2k\pi - \frac{1}{2}\pi\right) = -\frac{\pi}{4} + k\pi. \tag{F}$$

Equations E and F may be combined into the single equation

$$\tan^{-1}\frac{n^2}{2.1^2}+\tan^{-1}\frac{n^2}{2.2^2}+\tan^{-1}\frac{n^2}{2.3^2}+\ldots\ldots=\left(\frac{n}{2}-\frac{1}{4}\right)\pi, \qquad \text{(G)}$$

n being any positive integer.

Putting $\beta/\alpha=u$, equation D may be written more simply

$$\sum_{-\infty}^{+\infty}{}_{\nu}\tan^{-1}\frac{1}{2(\nu+u)^2}=\pi, \qquad \text{(D')}$$

u being any real quantity. It seems very remarkable that the value of this sum should be independent of u. And if, in formula C, we put $X=Y=n\alpha\frac{\pi}{2}$, n being any positive integer, we obtain

$$\sum_{-\infty}^{+\infty}{}_{\nu}\tan^{-1}\frac{n^2}{2(\nu+u)^2}\equiv 0 \pmod{\pi}.$$

The actual value of the series is $n\pi$. For the sum from $\nu=1$ to $\nu=+\infty$ does not exceed $\left(\frac{n}{2}-\frac{1}{4}\right)\pi$* (by equation G), the sum from $\nu=-2$ to $\nu=-\infty$ is less than $\left(\frac{n}{2}-\frac{1}{4}\right)\pi$, and the two omitted terms are together less than π; and on the other hand the sum from $\nu=0$ to $\nu=+\infty$ is at least as great as $\left(\frac{n}{2}-\frac{1}{4}\right)\pi$, and the sum from $\nu=-1$ to $\nu=-\infty$ is at least as great as $\left(\frac{n}{2}-\frac{1}{4}\right)\pi$. Hence the value of the series lies between $\left(n-\frac{1}{2}\right)\pi$ and $\left(n+\frac{1}{2}\right)\pi$; but it is an exact multiple of π; therefore

$$\sum_{-\infty}^{+\infty}{}_{\nu}\tan^{-1}\frac{n^2}{2(\nu+u)^2}=n\pi, \qquad \text{(H)}$$

n being any positive integer. This is a generalization of equation D or D′; but also equation G, which was employed in the proof of H, is itself an obvious corol-

*In the reasoning, u is supposed non-negative; obviously the argument is precisely similar if u is supposed non-positive.

lary from H, so that this last equation may be said to comprise all the preceding arithmetical equations $(D - G)$.

Finally, we remark that formula C shows that the series

$$\sum_{-\infty}^{+\infty} {}_{\nu} \tan^{-1} \frac{a}{(\nu + u)^2}$$

is not independent of u except when a is half the square of an integer;* and it thus seems all the more remarkable that it should be independent of u in that case. The value of this series (obtained by putting $X = Y = \sqrt{2a}\, a \frac{\pi}{2}$ in formula C) is given by

$$\sum_{-\infty}^{+\infty} {}_{\nu} \tan^{-1} \frac{a}{(\nu + u)^2}$$

$$= - \tan^{-1} \frac{\tanh \sqrt{2a}\, \frac{\pi}{2}}{\tan \left(\sqrt{2a} - 2u\right) \frac{\pi}{2}} - \tan^{-1} \frac{\tanh \sqrt{2a}\, \frac{\pi}{2}}{\tan \left(\sqrt{2a} + 2u\right) \frac{\pi}{2}} + \lambda \pi,$$

λ being an integer so chosen that the result shall lie between $E(\sqrt{2a}).\pi$ and $[E(\sqrt{2a}) + 1]\, \pi$, since the series is evidently intermediate in value between those which would arise on writing instead of a the two half-squares between which it lies.

The orthogonal trajectories of the preceding system of curves are given by the equation

$$\sin x\, dx + \sin y\, dy = 0;$$

these curves have the property

$$- \vartheta_N = \vartheta_0 + \vartheta_1 + \vartheta_{-1} + \vartheta_2 + \vartheta_{-2} + \vartheta_3 + \vartheta_{-3} + \dots,$$

and their equation is

$$\cos x + \cos y = c \quad \text{or} \quad \cos X \cosh Y = C.$$

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* It is, of course, periodic in respect to u in any case, the period being 1.

The curves having the property

$$\vartheta_T = \vartheta_0 + \vartheta_1 + \vartheta_{-1} + \vartheta_2 + \vartheta_{-2} + \vartheta_3 + \vartheta_{-3} + \ldots .$$

are given by the differential equation

$$dx/\sin x = dy/\sin y;$$

the solution of this is

$$\sin\frac{1}{2}(x+y) = c\sin\frac{1}{2}(x-y) \quad \text{or} \quad \sin X = C\sinh Y.$$

This last equation is otherwise evident from the consideration that the curve now required is so defined that its angle of intersection with any curve of the system first considered in this section shall be bisected by a parallel to the axis of X. Hence, the equation of that system having been found to be $\sin X \sinh Y = C$, it is plain that the required curves are given by $\sin X = C \sinh Y$.

On Rotations in Space of Four Dimensions.

By F. N. Cole.

1.

Linear Configuration in Four-Dimensional Space.

1. In a four-dimensional space of constant zero curvature, suppose any point to be selected, and through this point four linear solids mutually at right angles to each other, to be drawn. Taken in pairs, these four solids intersect in six planes, and taken by threes they intersect in four straight lines. Taken all together, they have only the selected point in common. This system of the point, the four lines, the six planes, and the four solids may be employed as a coordinate configuration. The point will be the origin, the four lines may be called the axes of x, y, z and w, the six planes the planes of xy, xz, xw, yz, yw and zw, the four linear solids, the solids of xyz, xyw, xzw and yzw respectively. The coordinate solids, in the order as written, are defined by the equations

$$w=0,\ z=0,\ y=0,\ x=0;$$

the planes by

$$\begin{array}{cccccc} w=0, & w=0, & z=0, & x=0, & x=0, & x=0, \\ z=0, & y=0, & y=0, & w=0, & z=0, & y=0, \end{array}$$

and the axes by

$$\begin{array}{cccc} y=0, & x=0, & x=0, & x=0, \\ z=0; & z=0, & y=0, & y=0, \\ w=0, & w=0, & w=0, & z=0. \end{array}$$

In pairs, the four axes determine the six coordinate planes; in threes, they determine the four coordinate solids. The six planes taken in pairs would seem at first sight to intersect in fifteen straight lines. But the two planes xy and zw have evidently no element in common except the origin, since this is the only point for which we can have simultaneously $w=0$, $z=0$, $x=0$, $y=0$. The same is evidently true for any pair of planes whose symbols have no letter in common. There are three such pairs, and accordingly the fifteen apparent lines of intersection reduce to twelve. Again, the pairs of planes xy and xz, xy and

yz, xz and yz, all intersect on the line $x=0$, $y=0$, $z=0$. All the pairs of planes can be similarly arranged, in four sets of three each, which have the same line of intersection. The twelve lines therefore reduce to the four coordinate axes, each counted three times.

Of the thirty combinations of the coordinate planes by threes, four sets which have each a common line have just been considered. Beside these, the four sets in each of which the same letter occurs three times in the symbols for the three planes, determine each a solid corresponding to the repeated letter. Thus the planes xy, xz and xw determine the solid $x=0$. The remaining twenty-two sets of three planes determine no elementary configuration.

The numerical arrangement of the parts of the coordinate configuration may therefore be briefly expressed as follows:

Each coordinate solid contains three coordinate planes and three coordinate axes. Each coordinate plane is contained in two coordinate solids and contains two coordinate axes. Each coordinate axis is contained in three coordinate solids and in three coordinate planes.

The coordinate (rectangular) configurations of the ordinary plane and space geometries divide the angular space about the origin into four plane right angles and eight right triedral angles respectively.

Similarly in space of four dimensions the angular space about the origin is divided by the coordinate configuration just discussed into sixteen right tetraedral angles.

2. A point in four-dimensional space is determined by four coordinates referred to the four coordinate solids. An equation of the first degree between these four coordinates defines a linear three-dimensional configuration. Such a configuration I shall call a *lineoid.* Among the lineoids the coordinate solids are of course included.

Geometrically a lineoid is determined by four points. For although four points are determined by sixteen constants, each point has, while remaining within the lineoid, still three degrees of freedom. Of the sixteen constants there remain therefore only 16—12, or 4, and this number corresponds to the four essential constants in the equation of the lineoid.

Two equations between the four coordinates analytically define a plane. But if two lineoids intersect in a given plane, any pair of linear combination of the two lineoids will intersect in the same plane. Two such pairs of linear combinations involve two essential constants. These subtracted from the eight

constants which determine the two lineoids leave six constants which determine their plane of intersection. Geometrically a plane is determined by three points. For each of the three points has, while remaining in the plane, two degrees of freedom, so that we have $3 \times 2 = 6$ essential constants.

Similarly three linear equations or two points determine a straight line. The two points involve eight constants, but as each has one degree of freedom on the line, these eight reduce to six essential constants.

The linear configurations in four-dimensional space are therefore determined as follows:

A lineoid by 4 constants.
A plane by 6 constants.
A line by 6 constants.
A point by 4 constants.

The dualistic relation between the lineoids and points on the one hand and the planes and lines on the other appears clearly in this table.

3. Among these linear configurations we are especially interested in those which contain the origin, and for which accordingly the constant terms in all the defining equations disappear. In this case a lineoid is determined by three further conditions, a plane by four, and a line by three. The geometry of these configurations is therefore analogous in form to that of the planes, lines and points of ordinary three-dimensional space.

As, however, the greater portion of the following developments hold for all linear configurations, whether they include the origin or not, I shall state them for the most part in their full generality.

If the equation of a lineoid be

$$ax + by + cz + dw + e = 0,$$

then we may write $\dfrac{a}{\sqrt{a^2+b^2+c^2+d^2}} = \cos\alpha$, $\dfrac{b}{\sqrt{a^2+b^2+c^2+d^2}} = \cos\beta$, $\dfrac{c}{\sqrt{a^2+b^2+c^2+d^2}} = \cos\gamma$, $\dfrac{d}{\sqrt{a^2+b^2+c^2+d^2}} = \cos\delta$, where the four cosines are connected by the identity $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 1$.

These four cosines I call the direction cosines of the corresponding lineoid. The quantity $\dfrac{e}{\sqrt{a^2+b^2+c^2+d^2}}$ may be called the perpendicular distance from the origin on the given lineoid.

If the direction cosines of two lineoids be

$$\cos\alpha\,,\ \cos\beta\,,\ \cos\gamma\,,\ \cos\delta\,,$$
$$\cos\alpha',\ \cos\beta',\ \cos\gamma',\ \cos\delta'$$

respectively, we may form from these the six determinants of the second order:

$$\begin{array}{ll}\cos\alpha\cos\beta'-\cos\beta\cos\alpha' & \cos\beta\cos\gamma'-\cos\gamma\cos\beta',\\ \cos\alpha\cos\gamma'-\cos\gamma\cos\alpha' & \cos\beta\cos\delta'-\cos\delta\cos\beta',\\ \cos\alpha\cos\delta'-\cos\delta\cos\alpha' & \cos\gamma\cos\delta-\cos\delta\cos\gamma'.\end{array}$$

The six quantities obtained by dividing each of these by the square root of the sum of the square of all of them I denote by P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}, and these I call the six direction cosines of the given planes. Thus

$$P_{12}=\frac{\cos\alpha\cos\beta'-\cos\beta\cos\alpha'}{\sqrt{1-(\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma'+\cos\delta\cos\delta')^2}}.$$

The denominator of the P's can only vanish when $\alpha=\alpha'$, $\beta=\beta'$, $\gamma=\gamma'$, $\delta=\delta'$, in which case the two given lineoids coïncide in all but their constant terms.

Beside the 6 P's the plane has also for determining elements the perpendiculars from the origin on the two given lineoids.

Between the P's there is the identity

$$P_{12}^2+P_{13}^2+P_{14}^2+P_{23}^2+P_{24}^2+P_{34}^2=1. \tag{1}$$

But since a plane through the origin* is determined by four constants, there must be still another identity connecting the P's. This is

$$P_{12}P_{34}+P_{13}P_{42}+P_{14}P_{23}=0 \tag{2}$$

as is readily seen from the development of the identically vanishing determinant

$$\begin{vmatrix}\cos\alpha, & \cos\beta, & \cos\gamma, & \cos\delta\\ \cos\alpha', & \cos\beta', & \cos\gamma', & \cos\delta'\\ \cos\alpha, & \cos\beta, & \cos\gamma, & \cos\delta\\ \cos\alpha', & \cos\beta', & \cos\gamma', & \cos\delta'\end{vmatrix}.$$

4. If we multiply the 6 P's by an arbitrary quantity K, we may regard the resulting quantities as the 6 *homogeneous* coordinates of a plane through the origin. These six coordinates are then identical with the six Plücker coordinates of a line in three-dimensional space. The geometry of planes through the origin in four-dimensional space is therefore exactly parallel to the Plücker line geometry, and every proposition of the one theory can be transferred to the other, so far as the geometrical distinction between the three- and the four-dimensional spaces interposes no obstacle. The examination of this correspondence in detail I intend to treat in a future paper. For the present I will merely estab-

* In treating the direction cosines alone we may, of course, suppose the given plane to pass through the origin.

lish two fundamental propositions for the geometry of the planes of a four-dimensional space.

5. Two lineoids,
$$ax + by + cz + dw + e = 0,$$
$$a'x + b'y + c'z + d'w + e' = 0$$

I call perpendicular to each other, if $aa' + bb' + cc' + dd' = 0$, or in terms of the direction cosines of the two lineoids,

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' + \cos\delta\cos\delta' = 0.$$

If we have two pair of lineoids such that both lineoids of the one pair are perpendicular to both of the other pair, the planes of intersection of each pair are naturally called perpendicular to each other. Two such planes will have no line in common. I call two such pairs of planes *absolutely* perpendicular to each other. Thus any pair of coordinate planes whose symbols contain no common letter, as xy and zw, are absolutely perpendicular to each other.

6. We can now establish at once the important proposition: Through a given point in a given plane only one plane can be passed which is absolutely perpendicular to the given plane. That this is the case is indicated by the fact that four conditions are requisite for absolute perpendicularity, and these are sufficient to determine the four essential constants of the second plane. We verify the proposition by the actual determination of the six direction cosines of the second plane in terms of those of the first.

We distinguish the different lineoids by subscripts 1, 2, 3, 4, and suppose that the planes (12) and (34) to be the first and second planes respectively. The four equations of conditions are then

$$\cos\alpha_1\cos\alpha_3 + \cos\beta_1\cos\beta_3 + \cos\gamma_1\cos\gamma_3 + \cos\delta_1\cos\delta_3 = 0,$$
$$\cos\alpha_1\cos\alpha_4 + \cos\beta_1\cos\beta_4 + \cos\gamma_1\cos\gamma_4 + \cos\delta_1\cos\delta_4 = 0,$$
$$\cos\alpha_2\cos\alpha_3 + \cos\beta_2\cos\beta_3 + \cos\gamma_2\cos\gamma_3 + \cos\delta_2\cos\delta_3 = 0,$$
$$\cos\alpha_2\cos\alpha_4 + \cos\beta_2\cos\beta_4 + \cos\gamma_2\cos\gamma_4 + \cos\delta_2\cos\delta_4 = 0.$$

From these we readily deduce

$$P_{12}\cos\beta_3 + P_{13}\cos\gamma_3 + P_{14}\cos\delta_3 = 0,$$
$$P_{12}\cos\beta_4 + P_{13}\cos\gamma_4 + P_{14}\cos\delta_4 = 0,$$
$$P_{12}\cos\alpha_3 - P_{23}\cos\gamma_3 - P_{24}\cos\delta_3 = 0,$$
$$P_{12}\cos\alpha_4 - P_{23}\cos\gamma_4 - P_{24}\cos\delta_4 = 0,$$
$$P_{13}\cos\alpha_3 + P_{23}\cos\beta_3 - P_{34}\cos\delta_3 = 0,$$
$$P_{13}\cos\alpha_4 + P_{23}\cos\beta_4 - P_{34}\cos\delta_4 = 0,$$
$$P_{14}\cos\alpha_3 + P_{24}\cos\beta_3 + P_{34}\cos\gamma_3 = 0,$$
$$P_{14}\cos\alpha_4 + P_{24}\cos\beta_4 + P_{34}\cos\gamma_4 = 0.$$

And from these again, if we distinguish the P's of the second plane by accents, we have

$$\begin{aligned}
&P_{12}P'_{23} - P_{14}P'_{34} = 0 && P'_{13} = -\frac{P_{34}}{P_{12}}P'_{34},\\
&P_{12}P'_{24} + P_{13}P'_{34} = 0 && P'_{14} = \frac{P_{23}}{P_{12}}P'_{34},\\
&P_{12}P'_{13} + P_{24}P'_{34} = 0 \;\therefore && P'_{23} = \frac{P_{14}}{P_{12}}P'_{34},\\
&P_{12}P'_{14} - P_{23}P'_{34} = 0 && P'_{24} = -\frac{P_{13}}{P_{12}}P'_{34},\\
&P_{13}P'_{12} + P_{34}P'_{24} = 0 && P'_{12} = -\frac{P_{34}}{P_{13}}P'_{24} = \frac{P_{34}}{P_{12}}P'_{34}.
\end{aligned}$$

If now we write $$P'_{12} = KP_{34}$$

we have
$$\begin{aligned}
P'_{13} &= KP_{42},\\
P'_{14} &= KP_{23},\\
P'_{23} &= KP_{14},\\
P'_{24} &= KP_{31},\\
P'_{34} &= KP_{12}.
\end{aligned}$$

But since $\Sigma P^2_{ik} = 1$ and $\Sigma P'^2_{ik} = 1$, we have $K^2 = 1$, $K = \pm 1$.

We may take either of the values of K. If we take $K = +1$ we have

$$\begin{aligned}
P'_{12} &= P_{34},\\
P'_{13} &= P_{42},\\
P'_{14} &= P_{23},\\
P'_{23} &= P_{14},\\
P'_{24} &= P_{31},\\
P'_{34} &= P_{12}.
\end{aligned} \tag{3}$$

As a final verification we have

$$P'_{12}P'_{34} + P'_{13}P'_{42} + P'_{14}P'_{23} = P_{34}P_{12} + P_{42}P_{13} + P_{23}P_{14} = 0.$$

The equations (3), regarded as defining the transition from the given plane to its absolute perpendicular plane, are equivalent in the Plücker geometry to the analytic definition of a dualistic transformation. The relation between the planes P and P' is evidently a reciprocal one.

7. These equations may also be simply interpreted within the four-dimensional space as follows: We have regarded a plane as the intersection of two lineoids. We may also regard it as determined by two straight lines. For convenience we will suppose the plane and its determining elements all to pass through the

origin. If the coordinates of any point on any one of the determining lines be x, y, z, w, we may call the quantities

$$\frac{x}{\sqrt{x^2+y^2+z^2+w^2}}, \quad \frac{y}{\sqrt{x^2+y^2+z^2+w^2}}, \quad \frac{z}{\sqrt{x^2+y^2+z^2+w^2}}, \quad \frac{w}{\sqrt{x^2+y^2+z^2+w^2}}$$

the direction cosines of the line. From the direction cosines of the two determining lines we may then form determinants P' of the second order as before in the case of the lineoids.

If a line and a lineoid have the same direction cosines they may be called perpendicular to each other. It appears at once that if a plane be determined by two lineoids and a second plane by two lines whose direction cosines are respectively equal to those of the lineoid, these two planes are absolutely perpendicular to each other. If, therefore, the direction cosines of a plane as determined by two lineoids be P_{ik} and those of the same plane as determined by two lines be P'_{ik}, the equations (3) hold between these two sets of coordinates. In other words, the equations (3) define the transformation from a lineoid geometry to a line geometry.

These results can of course be easily verified analytically by expressing the direction cosines of a line in terms of the direction cosines of the three lineoids which intersect in the line.

8. The condition that two planes shall have a common line is also readily obtained. Thus, if the two planes be determined as before as the intersection of the lineoids 1 and 2, 3 and 4, each of these lineoids must contain the common line of the two planes. The condition for this is obviously

$$\begin{vmatrix} \cos\alpha_1, & \cos\beta_1, & \cos\gamma_1, & \cos\delta_1 \\ \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2, & \cos\delta_2 \\ \cos\alpha_3, & \cos\beta_3, & \cos\gamma_3, & \cos\delta_3 \\ \cos\alpha_4, & \cos\beta_4, & \cos\gamma_4, & \cos\delta_4 \end{vmatrix} = 0.$$

Expanding this determinant in quadratic minors, we have at once

$$P_{12}P'_{34} + P_{13}P'_{42} + P_{14}P'_{23} + P_{23}P'_{14} + P_{42}P'_{13} + P_{34}P'_{12} = 0. \tag{4}$$

This is identical with the Plücker condition that two straight lines in ordinary space shall have a common point.

9. If a plane P' have a line in common with a given plane P, and also with the plane absolutely perpendicular to P, we have the two equations of condition

$$P_{12}P'_{34} + P_{13}P'_{42} + P_{14}P'_{23} + P_{23}P'_{14} + P_{42}P'_{13} + P_{34}P'_{12} = 0,$$
$$P_{12}P'_{12} + P_{13}P'_{13} + P_{14}P'_{14} + P_{23}P'_{23} + P_{24}P'_{24} + P_{34}P'_{34} = 0.$$

From the symmetry of these equations it is at once evident the plane P has a line in common not only with P' but also with the plane absolutely perpendicular to P'. The relation between the two planes is therefore a reciprocal one. Two such planes I call *simply* perpendicular to each other. There are, therefore, ∞^2 planes simply perpendicular to a given plane. The situation of these planes can be readily understood by the aid of the following consideration: Through the point of intersection of the planes P and P' ∞^1 straight lines can be drawn in each plane. Any two of these lines lying one in the plane P, the other in the plane P', determine one of the simply perpendicular planes. Through each of the lines in P there, therefore, pass ∞^1 of the simple perpendicular planes, and these cut the plane P' in the bundle of rays through the intersection P and P'. It appears, therefore, that there are ∞^1 planes simply perpendicular to a given plane and cutting it in a given line. For example, the planes xz and yz, or any one of their ∞^1 linear combinations, are simply perpendicular to the plane xy.

In the development of this part of the subject I have not attempted anything like an exhaustive treatment. I have simply aimed to establish systematically so much of the theory of the linear configurations containing the origin in space of four dimensions as is of immediate use in the theory of rotations.

2.

THE GENERAL THEORY OF ROTATION IN FOUR-DIMENSIONAL SPACE.

1. The general collineation in space of four dimensions is defined by the four equations

$$
\begin{aligned}
x' &= \frac{a_1x + b_1y + c_1z + d_1w + e_1}{a_5x + b_5y + c_5z + d_5w + e_5}, \\
y' &= \frac{a_2x + b_2y + c_2z + d_2w + e_2}{a_5x + b_5y + c_5z + d_5w + e_5}, \\
z' &= \frac{a_3x + b_3y + c_3z + d_3w + e_3}{a_5x + b_5y + c_5z + d_5w + e_5}, \\
w' &= \frac{a_4x + b_4y + c_4z + d_4w + e_4}{a_5x + b_5y + c_5z + d_5w + e_5},
\end{aligned}
\tag{1}
$$

These involve twenty-four essential constants; that is, there are ∞^{24} possible collineations. That these form a group is clear. We are interested in the subgroup which converts a solid of the second order, more particularly a solid sphere, into itself. The equation of such a surface contains fourteen essential constants. Since these are to remain unchanged, we have fourteen equations

of condition among the twenty-four constants of the general collineation. The desired subgroup contains, therefore, ∞^{10} distinct operations.

This subgroup contains again a further subgroup composed of the rotations of the sphere. As these rotations I define those collineations of the four-dimensional space which not only convert the sphere into itself but also leave its center, assumed to be at the origin, and with it its polar solid plane, the lineoid at infinity, unchanged.* From this definition it appears at once that the denominator in the equations (1) reduce to a single constant term which may be regarded as combined with the coefficients of the numerators, and that the constant terms in the numerators reduce to zero.

The equations for the subgroup of these rotations therefore are of the form

$$\begin{aligned} x' &= a_1x + b_1y + c_1z + d_1w, \\ y' &= a_2x + b_2y + c_2z + d_2w, \\ z' &= a_3x + b_3y + c_3z + d_3w, \\ w' &= a_4x + b_4y + c_4z + d_4w. \end{aligned} \tag{2}$$

If the equation of the invariant sphere be $x^2+y^2+z^2+w^2=1$, the coefficients a, b, c, d are further connected by the ten equations of condition

$$\begin{aligned} &a_1^2+a_2^2+a_3^2+a_4^2=1, \qquad && a_1a_2+b_1b_2+c_1c_2+d_1d_2=0, \\ &b_1^2+b_2^2+b_3^2+b_4^2=1, && a_1a_3+b_1b_3+c_1c_3+d_1d_3=0, \\ &c_1^2+c_2^2+c_3^2+c_4^2=1, && a_1a_4+b_1b_4+c_1c_4+d_1d_4=0, \\ &d_1^2+d_2^2+d_3^2+d_4^2=1, && a_2a_3+b_2b_3+c_2c_3+d_2d_3=0, \\ & && a_2a_4+b_2b_4+c_2c_4+d_2d_4=0, \\ & && a_3a_4+b_3b_4+c_3c_4+d_3d_4=0. \end{aligned} \tag{3}$$

The equation (2) contains sixteen constants, and as these are connected by the ten relations (3), it appears that the group of rotation of a four-dimensional space about any fixed point contains ∞^6 distinct operations.

2. The theory of orthogonal transformation has been extensively studied by Cayley,† who has given a general method of expressing the n^2 coefficients of such a transformation in terms of the $\frac{1}{2}n(n-1)$ independent constants of the transformation. In the case of a four-dimensional space, if we call the six independent constants a, b, c, f, g, h, we have

* See also §9.

† Crelle XXXII.

$$
\left.\begin{aligned}
Ba_1 &= 1 - \vartheta^2 + f^2 - a^2 + g^2 - b^2 + h^2 - c^2, \\
Ba_2 &= 2(-a - f\vartheta + cg - bh), \\
Ba_3 &= 2(-b - cf - \vartheta g + ah), \\
Ba_4 &= 2(-c + bf - ag - h\vartheta), \\[4pt]
Bc_1 &= 2(b + g\vartheta - cf + ah), \\
Bc_2 &= 2(h + fg + c\vartheta - ab), \\
Bc_3 &= 1 - \vartheta^2 + g^2 - b^2 + c^2 - h^2 + a^2 - f^2, \\
Bc_4 &= 2(-f + gh - bc - a\vartheta), \\[4pt]
Bb_1 &= 2(a + f\vartheta - bh + cg), \\
Bb_2 &= 1 - \vartheta^2 + f^2 - a^2 + b^2 - g^2 + c^2 - h^2, \\
Bb_3 &= 2(-h + fg - ab - c\vartheta), \\
Bb_4 &= 2(g + fh + b\vartheta - ac), \\[4pt]
Bd_1 &= 2(c + h\vartheta - ag + bf), \\
Bd_2 &= 2(-g + hf - ac - b\vartheta), \\
Bd_3 &= 2(f + gh + a\vartheta - bc), \\
Bd_4 &= 1 - \vartheta^2 + h^2 - c^2 + a^2 - f^2 + b^2 - g^2.
\end{aligned}\right\} \tag{4}
$$

where $\qquad \vartheta = af + bg + ch$

and $\qquad B = 1 + a^2 + b^2 + c^2 + g^2 + f^2 + h^2 + \vartheta^2.$

The question now arises, and this leads to developments of fundamental importance, whether a rotation in four-dimensional space as defined by the equation (2) and (3) or (4) has any points other than the origin fixed, as is the case in the corresponding problem in three-dimensional space. If there be such points, they will be determined by putting in equations (3) x', y', z', w', equal respectively to x, y, z, w and solving the resulting equation,

$$
\left.\begin{aligned}
(a_1 - 1)\,x + b_1 y + c_1 z + d_1 w &= 0, \\
a_2 x + (b_2 - 1)\,y + c_2 z + d_2 w &= 0, \\
a_3 x + b_3 y + (c_3 - 1)\,z + d_3 w &= 0, \\
a_4 x + b_4 y + c_4 z + (d_4 - 1)\,w &= 0.
\end{aligned}\right\} \tag{5}
$$

If these equations have a common solution other than $0, 0, 0, 0$, we must have

$$
\begin{vmatrix}
a_1 - 1 & b_1 & c_1 & d_1 \\
a_2 & b_2 - 1 & c_2 & d_2 \\
a_3 & b_3 & c_3 - 1 & d_3 \\
a_4 & b_4 & c_4 & d_4 - 1
\end{vmatrix} \equiv 0.
$$

Substituting now for the a, b, c, d their values as given by equation (4), we find that this determinant is *not* identically 0, but reduces to

$$\frac{\vartheta^2}{B}.*$$

In order, therefore, that a rotation may have any point other than the origin fixed, we must have

$$\vartheta = 0.$$

We must, therefore, divide the rotations of a four-dimensional space about a fixed point into two classes, one containing ∞^5 distinct operations, each of which leaves other points beside the origin fixed, and the other containing the remaining ∞^6 distinct operations which do not possess this property.

If ϑ be 0, the four equations (5) apparently reduce to three independent ones, which accordingly determine a fixed line. But this is not all. The equations (5) in this case really reduce to two, which accordingly determine a fixed plane. That this is so is readily seen if we write in the four equations in the place of the a, b, c, d their value as given in equation (4), in which we are now to put $\vartheta = 0$.

We have then

$$\left.\begin{aligned}
(-a^2-b^2-c^2)\,x + (a - bh + cg)\,y + (b - cf + ah)\,z + (c - ag + bf)\,w &= 0,\\
(-a + cg + bh)\,x + (-a^2-g^2-h^2)\,y + (h + fg - ab)\,z + (-g + hf - ac)\,w &= 0,\\
(-b - cf + ah)\,x + (-h + fg - ab)\,y + (-b^2-f^2-h^2)\,z + (f + gh - bc)\,w &= 0,\\
(-c + bf - ag)\,x + (g + fh - ac)\,y + (-f + gh - bc)\,z + (-c^2-f^2-g^2)\,w &= 0.
\end{aligned}\right\} \quad (6)$$

If, now, we multiply the second, third, and fourth equations by a, b, and c respectively and add the results to the first equation, remembering that $af + bg + ch = 0$, the resulting coefficients all vanish identically. Moreover, if we multiply the second, third, and fourth equations by f, g, and h respectively and add them together, the resulting coefficients again all vanish.

We have, then, this result: *Of the ∞^6 rotations in general defined by equations* (2) *and* (3), *only a minor class of ∞^5 rotations leave any point in space except the origin fixed. Each of these ∞^5 rotations leaves an entire plane fixed.*

We shall find, however, that these latter ∞^5 rotations do not constitute a group. The resultant of two such rotations does not in general leave any point except the origin fixed.

4. Those rotations which leave a plane fixed, I shall hereafter call "simple" rotations. The fixed plane for such a rotation, it must be noted, is not only fixed

---

* See Scott's Determinants, p. 233.

in the sense that it is converted into itself, but it is fixed absolutely, i. e. every point in it is fixed by itself.

The plane absolutely perpendicular to the fixed plane is also evidently converted into itself, but its separate points do not remain fixed. Since the spherical solid is also converted into itself, it follows that the circle of intersection of the absolutely perpendicular plane with the sphere is likewise converted into itself. Again, every collineation of the four-dimensional space is also a collineation of any plane which it may leave fixed. It appears, therefore, that the absolutely perpendicular plane is simply rotated through a certain angle into itself. This angle we will call the angle of the rotation the four-dimensional space considered.

The ∞^2 planes which are simply perpendicular to the fixed plane of the "simple" rotation have a line in common with this plane and a line in common with the absolutely perpendicular plane. Of these two lines, the one in the fixed plane remains in every case fixed, while that in the absolutely perpendicular plane is rotated in that plane about the origin through the angle of the rotation. If among these planes we select those which have a given line of intersection with the absolutely fixed plane, the transformation which these undergo is exactly the same as that of planes through the axis of a rotation in three-dimensional space.

5. Returning, now, to the analytic treatment of the subject, we can at once, in case $\vartheta = 0$, find an interpretation for the six independent constants a, b, c, f, g, h. *These are proportional to the six direction cosines of the fixed plane.*

For the fixed plane is determined by any two of the lineoids defined by equation (6), say the first and second. Calculating the direction cosines of the plane of intersection of these lineoids, and remembering again that $af + bg + ch = 0$, we have at once the six quantities a^2, ab, ac, ah, $-ag$, af.

These we may regard as the six homogeneous coordinates of the fixed plane. Between these we have already the Plücker identity $af + bg + ch = 0$. To obtain the six quantities p, we have only to divide these six coordinates by the square root of the sum of their squares. Thus,

$$P_{12} = \frac{a}{B} \qquad P_{23} = \frac{h}{B},$$

$$P_{13} = \frac{b}{B} \qquad P_{24} = \frac{-g}{B},$$

$$P_{14} = \frac{c}{B} \qquad P_{34} = \frac{f}{B} \text{ where } B = \sqrt{1+a^2+b^2+c^2+f^2+g^2+h^2}.$$

For the absolutely perpendicular plane we have

$$P'_{12} = \frac{f}{B} \qquad P_{23} = \frac{c}{B},$$

$$P'_{13} = \frac{g}{B} \qquad P_{24} = \frac{-b}{B},$$

$$P'_{14} = \frac{h}{B} \qquad P_{34} = \frac{a}{B}.$$

Since the fixed plane is determined by four independent constants, while the a, b, c, f, g, h, with the identity $af + bg + ch = 0$, constitute a system of five independent quantities, it is clear that the *extent* of the rotation is measured by a single constant; that is, that the rotation is in itself one-dimensional, a result which agrees with the preceding geometrical consideration.

It remains to determine the extent of the rotation; that is, the angle mentioned above. If we write

$$a = \cos\alpha \tan\frac{\phi}{2}, \qquad b = \cos\beta \tan\frac{\phi}{2}, \quad c = \cos\gamma \tan\frac{\phi}{2},$$

$$f = \cos\delta \tan\frac{\phi}{2}, \quad -g = \cos\varepsilon \tan\frac{\phi}{2}, \quad h = \cos\delta \tan\frac{\phi}{2},$$

where the six cosines are the six direction cosines of the fixed plane, that is, the six P's, so that

$$B = \sqrt{1 + \tan^2\frac{\phi}{2}} = \sec\frac{\phi}{2},$$

the angle ϕ thus defined is the angle of the rotation.

It will be sufficient to prove this in a single case. We will, therefore, assume $b = c = f = g = h = 0$, so that the fixed plane shall be the plane of zw. From equation (4) we have, then, for the equation of the rotation,

$$x' = \frac{1 - a^2}{1 + a^2}x + \frac{2a}{1 + a^2}y \qquad z' = z,$$

$$y' = \frac{-2a}{1 + a^2}x + \frac{1 - a^2}{1 + a^2}y \qquad w' = w.$$

Comparing these with the equation for rotation about the origin in the plane xy,

$$x' = x\cos\phi - y\sin\phi,$$

$$y' = x\sin\phi + y\cos\phi,$$

we have

$$\cos\phi = \frac{1 - a^2}{1 + a^2}, \quad \sin\phi = \frac{-2a}{1 + a^2},$$

$$\therefore\ a^2 = \frac{1 - \cos\phi}{1 + \cos\phi} = \tan^2\frac{\phi}{2}.$$

6. I have already mentioned that the combination of two rotations with fixed planes does not in general give a rotation with a fixed plane. The discussion of this question is included in the general theory of the composition of rotation, to which I now proceed. In the following I obtain expressions for the quantities $a, b, c, f, g, h, \vartheta$ of a resultant rotation in terms of the same quantities for the two component rotations.

If we write the two component rotations in the form

$$x_i' = c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3 + c_{i4}x_4,$$
$$x_i'' = c'_{i1}x'_1 + c'_{i2}x'_2 + c'_{i3}x'_3 + c'_{i4}x'_4,$$

and suppose the rotations to occur in order as written, we have for the resultant rotation

$$x_i'' = c''_{i1}x_1 + c''_{i2}x_2 + c''_{i3}x_3 + c''_{i4}x_4,$$

where

$$c''_{ij} = c'_{i1}c_{1j} + c'_{i2}c_{2j} + c'_{i3}c_{3j} + c'_{i4}c_{4j}. \qquad (\alpha)$$

Turning now to equation (4), we have

$$\begin{aligned} B''(c''_{11} + c''_{22} + c''_{33} + c''_{44}) &= 1 - \vartheta''^2 - a''^2 - b''^2 - c''^2 + f''^2 + g''^2 + h''^2 \\ &+ 1 - \vartheta''^2 - a''^2 + b''^2 + c''^2 + f''^2 - g''^2 - h''^2 \\ &+ 1 - \vartheta''^2 + a''^2 + b''^2 + c''^2 - f''^2 + g''^2 - h''^2 \\ &+ 1 - \vartheta''^2 + a''^2 + b''^2 - c''^2 - f''^2 - g''^2 + h''^2 \\ &= 4(1 - \vartheta''^2). \end{aligned}$$

By the aid of equation (α) we then have at once

$$\begin{aligned} &\frac{4BB'}{B''}(1 - \vartheta''^2) \\ &= (1 - \vartheta'^2 - a'^2 - b'^2 - c'^2 + f'^2 + g'^2 + h'^2)(1 - \vartheta^2 - a^2 - b^2 - c^2 + f^2 + g^2 + h^2) \\ &+ (1 - \vartheta'^2 - a'^2 + b'^2 + c'^2 + f'^2 - g'^2 - h'^2)(1 - \vartheta^2 - a^2 + b^2 + c^2 + f^2 - g^2 - h^2) \\ &+ (1 - \vartheta'^2 + a'^2 - b'^2 + c'^2 - f'^2 + g'^2 - h'^2)(1 - \vartheta^2 + a^2 - b^2 + c^2 - f^2 + g^2 - h^2) \\ &+ (1 - \vartheta'^2 + a'^2 + b'^2 - c'^2 - f'^2 - g'^2 + h'^2)(1 - \vartheta^2 + a^2 + b^2 - c^2 - f^2 - g^2 + h^2) \\ &+ 4(a' + f'\vartheta' - b'h' + c'g')(-a - f\vartheta + cg - bh) \\ &+ 4(b' + g'\vartheta' - c'f' + a'h')(-b - cf - g\vartheta + ah) \\ &+ 4(c' + h'\vartheta' - a'g' + b'f')(-c + bf - ag - h\vartheta) \\ &+ 4(-a' - f'\vartheta' + c'g' - b'h')(a + f\vartheta - bh + cg) \\ &+ 4(h' + f'g' + c'\vartheta' - a'b')(-h + fg - ab - c\vartheta) \\ &+ 4(-g' + h'f' - a'c' - b'\vartheta')(g + fh + b\vartheta - ac) \\ &+ 4(-b' - c'f' - g'\vartheta' + a'h')(b + g\vartheta - cf + ah) \\ &+ 4(-h' + f'g' - a'b' - c'\vartheta')(h + fg + c\vartheta - ab) \\ &+ 4(f' + g'h' + a'\vartheta' - b'c')(-f + gh - bc - a\vartheta) \\ &+ 4(-c' + b'f' - a'g' - h'\vartheta')(c + h\vartheta - ag + bf) \\ &+ 4(g' + f'h' + b'\vartheta' - a'c')(-g + hf - ac - b\vartheta) \\ &+ 4(-f' + g'h' - b'c' - a'\vartheta')(f + gh + a\vartheta - bc). \end{aligned}$$

From this we readily obtain by multiplying out and recombining the results,

$$\frac{BB'}{B''}(1-\vartheta''^2)=(1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh')^2$$
$$-(a'f+af'+b'g+bg'+c'h+ch'+\vartheta+\vartheta')^2.$$

Hence
$$\vartheta''^2=+\frac{B''}{BB'}(a'f+af'-b'g+bg'+c'h+ch'+\vartheta+\vartheta')^2$$
$$-\frac{B''}{BB'}(1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh')^2$$
$$+1.$$

Now, if the component rotations each leave a plane fixed, we have $\vartheta=\vartheta'=0$. And if, in addition, these two fixed planes have a line in common, we have also $a'f+af'+b'g+bg'+c'h+ch'=0$. But in this case the resultant rotation will also leave the line of intersection of the two planes fixed and consequently will leave a whole plane fixed. We have therefore for this case

$$\vartheta''=0,$$
$$\frac{B''}{BB'}(1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh')^2=1,$$
$$\therefore\ B''=\frac{BB'}{(1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh')^2}$$
$$=\frac{(1+a^2+b^2+c^2+f^2+g^2+h^2+\vartheta^2)(1+a'^2+b'^2+c'^2+f'^2+g'^2+h'^2+\vartheta'^2)}{(1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh')}.$$

But since the a, b, c, f, g, h are independent variables, only connected in the present case by the three equations of condition

$$\vartheta=0,\ \vartheta'=0,\ a'f+af'+b'g+bg'+c'h+ch'=0,$$

it is clear that we have always

$$B''=\frac{BB''}{(1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh')^2}$$

and consequently

$$\vartheta''^2=\frac{B''}{BB'}(a'f+af'+b'g+bg'+c'h+ch'+\vartheta+\vartheta')^2$$
$$=\frac{(a'f+af'+b'g+bg'+c'h+ch'+\vartheta+\vartheta')^2}{(1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh')^2},$$
$$\vartheta''=\frac{a'f+af'+b'g+bg'+c'h+ch'+\vartheta+\vartheta'}{1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh'}.\text{*} \qquad (\beta)$$

*In the extraction of the square root the plus sign must be taken, as any simple example will show.

Again, from equation (4) we have

$$\begin{aligned}
B''(c''_{12}-c''_{21}) &= 4(a''+f''\vartheta''), & B''(c''_{34}-c''_{43}) &= 4(f''+a''\vartheta''),\\
B''(c''_{13}-c''_{31}) &= 4(b''+g''\vartheta''), & B''(c''_{42}-c''_{24}) &= 4(g'+b''\vartheta''),\\
B''(c''_{14}-c''_{41}) &= 4(c''+h''\vartheta''), & B''(c''_{23}-c''_{32}) &= 4(h''+c''\vartheta'').
\end{aligned}$$

From these, by the aid of the equations (α), we obtain

$$\begin{aligned}
4\frac{BB'}{B''}(a''+f''\theta'') = {} & 2(1-\theta'^2-a'^2-b'^2-c'^2+f'^2+g'^2+h'^2)(a+f\vartheta-bh+cg)\\
& + 2(a'+f'\vartheta-b'h+c'g)(1-\theta^2-a^2+b^2+c^2+f^2-g^2-h^2)\\
& + 4(b'+g'\vartheta'-c'f'+a'h')(-h+fg-ab-c\theta)\\
& + 4(c'+h'\vartheta'-a'g'+b'f')(g+fh+b\vartheta-ac)\\
& - 2(-a'-f'\vartheta'+c'g'-b'h')(1-\theta^2-a^2-b^2-c^2+f^2+g^2+h^2)\\
& - 2(1-\vartheta'^2-a'^2+b'^2+c'^2+f'^2-g'^2-h'^2)(-a-f\vartheta+cg-bh)\\
& - 4(h'+f'g'+c'\vartheta'-a'b')(-b-cf-g\vartheta+ah)\\
& - 4(-g'+h'f'-a'c'-b'\vartheta')(-c+bf-ag-h\vartheta),\\
4\frac{BB'}{B''}(f''+a''\vartheta'') = {} & 4(-c'+b'f'-a'g'-h'\vartheta')(b+g\vartheta-cf+ah)\\
& + 4(g'+f'h'+b'\vartheta'-a'c')(h+fg+c\vartheta-ab)\\
& + 2(-f'+g'h'-b'c'-a'\vartheta')(1-\vartheta^2+a^2-b^2+c^2-f^2+g^2-h^2)\\
& + 2(1-\vartheta'^2+a'^2+b'^2-c'^2-f'^2-g'^2-h'^2)(-f+gh-bc-a\theta)\\
& + 4(-b'-c'f'-g'\vartheta'+a'h')(c+h\vartheta-ag+bf)\\
& + 4(-h'+f'g'-a'b'-c'\vartheta')(-g+hf-ac-b\theta)\\
& + 2(1-\vartheta^2+a'^2-b'^2+c'^2-f'^2+g'^2-h'^2)(f+gh-ac-b\vartheta)\\
& + 2(f'+g'h'+a'\vartheta'-b'c')(1-\vartheta^2+a^2+b^2-c^2-f^2-g^2+h^2).
\end{aligned}$$

Combining these we obtain, after a series of easy reductions,

$$\begin{aligned}
& (a''+f'')(1+\vartheta'')(1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-hh'-gg')^2\\
&= (1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh'+\vartheta+\vartheta'\\
&\quad + a'f+af'+b'g+bg'+c'h+ch')(a+a'+f+f'-(f'+a')\vartheta-(f+a)\vartheta'\\
&\quad + bh'-b'h+c'g-cg'+bc'-b'c+gh'-g'h).
\end{aligned}$$

But from equation (β)

$$1+\vartheta''=\frac{1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh'+\vartheta+\vartheta'+c'f+cf'+b'g+bg'+c'h+ch'}{1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh'},$$

$$\therefore a''+f''=\frac{\begin{array}{l}a+a'+f+f'-(a+f)\vartheta'-(a'+f')\vartheta\\ \qquad\qquad +bh'-b'h+c'g-cg'+bc'-b'c+gh'-g'h\end{array}}{1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh'}.$$

Similarly

$$a''-f''=\frac{\begin{array}{r}a+a'-f-f'+(a'-f')\vartheta+(a-f)\vartheta' \\ +bh'-b'h+c'g-cg'+b'c-bc'+g'h-gh'\end{array}}{1-\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh'}.$$

Hence

$$a''=\frac{a+a'-f\vartheta'-f'\vartheta+bh'-b'h+c'g-cg'}{1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh'},$$

$$f''=\frac{f+f'-a'\vartheta-a\vartheta'+bc'-b'c+gh'-g'h}{1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh'}.$$

We can now write down at once all the formulae for the combination of two rotations. If we denote the common denominator $1+\vartheta\vartheta'-aa'-bb'-cc'-ff'-gg'-hh'$ by D, the formulae are

$$\begin{aligned}
Da''&=a+a'-f\vartheta'-f'\vartheta+bh'-b'h+c'g-cg', \\
Db''&=b+b'-g\vartheta'-g'\vartheta+cf'-c'f+a'h-ah', \\
Dc''&=c+c'-h\vartheta'-h'\vartheta+ag'-a'g+b'f-bf', \\
Df''&=f+f'-a\vartheta'-a'\vartheta+bc'-b'c+gh'-g'h, \\
Dg''&=g+g'-b\vartheta'-b'\vartheta+ca'-ac'+f'h-fh', \\
Dh''&=h+h'-c\vartheta'-c'\vartheta+ab'-a'b+fg'-f'g, \\
D\vartheta''&=\vartheta+\vartheta'+a'f+af'+b'g+bg'+c'h+ch'.
\end{aligned}$$

The formulae may be tested by aid of the equation

$$\vartheta''=a''f''+b''g''+c''h''.$$

It will be found that this condition is satisfied.

The condition that the resultant of two simple rotations shall be a simple rotation is now clear. We must have simultaneously $\vartheta''=0$, $\vartheta=0$, $\vartheta'=0$, which require $\qquad a'f+af'+b'g+bg'+c'h+ch'=0.$

That is, *the resultant of two simple rotations is itself a simple rotation when and only when the fixed planes of the component rotations have a line in common.* That this condition was *sufficient* was already clear from geometrical considerations. It now appears that it is also necessary.

7. I determine next the resultant of two simple rotations whose fixed planes are absolutely perpendicular to each other, the angles of the rotation being ϕ and ϕ' respectively, and K and K' denoting the $\tan\frac{\phi}{2}$ and $\tan\frac{\phi'}{2}$ respectively. If

the six direction cosines of the one plane be P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}, those of the other are P_{34}, P_{42}, P_{23}, P_{14}, P_{31}, P_{12}. We have then

$$\begin{aligned} a &= KP_{12}, & b &= KP_{13}, & c &= KP_{14}, & f &= KP_{34}, & g &= -KP_{24}, & h &= KP_{23}, \\ a' &= K'P_{34}, & b' &= K'P_{42}, & c' &= K'P_{23}, & f' &= K'P_{12}, & g' &= -K'P_{31}, & h' &= K'P_{14}. \end{aligned}$$

The equations of combination become in this case

$$\left.\begin{aligned} a'' &= \frac{a+a'}{1-aa'-bb'-cc'-ff'-gg'-hh'} = \frac{KP_{12}+K'P_{34}}{1-2KK'(P_{12}P_{34}+P_{13}P_{42}+P_{14}P_{23})} \\ &= KP_{12}+K'P_{34}, \\ b'' &= KP_{13}+K'P_{42}, \\ c'' &= KP_{14}+K'P_{23}, \\ f'' &= KP_{34}+K'P_{12}, \\ g'' &= KP_{24}+K'P_{31}, \\ h &= KP_{23}+K'P_{14}, \\ \vartheta &= KK'(P_{12}^2+P_{13}^2+P_{14}^2+P_{23}^2+P_{24}^2+P_{34}^2) \\ &= KK'. \end{aligned}\right\} \quad (\gamma)$$

8. If, now, a general rotation, $\vartheta'' \gtrless 0$, be given, we may decompose this into two simple rotations with fixed planes absolutely at right angles to each other by the aid of the above formulae (γ).

Squaring and adding the equations (γ) we have

$$\begin{aligned} a''^2+b''^2+c''^2+f''^2+g''^2+h''^2 = &\quad K^2(P_{12}^2+P_{13}^2+P_{14}^2+P_{23}^2+P_{24}^2+P_{31}^2) \\ &+ 2KK'(P_{12}P_{34}+P_{13}P_{42}+P_{14}P_{23}) \\ &+ \quad K'^2(P_{12}^2+P_{13}^2+P_{14}^2+P_{23}^2+P_{24}^2+P_{34}^2) \\ &= \quad K^2+K'^2. \end{aligned}$$

From this equation, in combination with the equation $\vartheta'' = KK'$, we can determine K and K'. These equations have four pairs of solutions. Any one of these being given, the others are deduced from it, (1), by interchanging the values of K and K', (2) by changing the signs of the values, and (3) by both interchanging the values and changing the signs.

Again, from the equation (γ), we obtain at once

$$\begin{aligned} P_{12} &= \frac{Ka''-K'f''}{K^2-K'^2} & P_{34} &= \frac{Kf''-K'a''}{K^2-K'^2}, \\ P_{13} &= \frac{Kb''-K'g''}{K^2-K'^2} & P_{42} &= \frac{Kg''-K'b''}{K^2-K'^2}, \\ P_{14} &= \frac{Kc''-K'h''}{K^2-K'^2} & P_{23} &= \frac{Kh''-K'c''}{K^2-K'^2}. \end{aligned}$$

From these equations it appears that if the signs of K and K' be changed, those of the P_{ik} are changed at the same time. In other words, such a change of signs does not affect the position of the fixed plane, but merely changes the point from which it is viewed from one side of the plane to the other.

Again, an interchange of K and K' converts the P_{ik} of the plane into those of the absolutely perpendicular plane.

Our system of solutions leads therefore to only one pair of fixed planes. We have accordingly the following proposition:

Every rotation of a four-dimensional space for which $\vartheta \neq 0$ can be reduced to a succession of two simple rotations whose fixed planes are absolutely perpendicular to each other. This decomposition can be effected in only one way. The quantity θ is the product of the tangents of half the angles of the simple rotations. The quantity

$$B = 1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2 + \vartheta^2$$
$$= (1 + K^2 + K'^2 + K^2K'^2) = (1 + K^2)(1 + K'^2),$$

i. e. *it is the product of the squares of the secants of the two angles of the two simple rotations.*

If we choose as axes of coordinates four lines lying two (x and y) in the fixed plane of the one simple component rotation, and two (z and w) in the fixed plane of the other, the equations of the rotation reduce evidently to the form

$$\begin{aligned} x' &= x\cos\phi - y\sin\phi, & z' &= z\cos\phi' - w\sin\phi', \\ y' &= x\sin\phi - y\cos\phi, & w' &= z\sin\phi' + w\cos\phi'. \end{aligned} \qquad (\delta)$$

The simple rotations occur when one of the angles ϕ, ϕ' is 0. In conformity with the use of the name simple rotation, the general rotation may be called a double rotation.

9. In closing, it remains to be noted that the orthogonal transformations defined by equations (2) and (3) include not only the ∞^6 rotations of the four-dimensional space about the origin, but also an equal number of other transformations which are most simply described as combinations of the preceding rotations with a *reflection* on any lineoid through the origin. For convenience we will call this second class of transformations the *conjugate* transformations.

For example, the transformation $x' = -x$, $y' = y$, $z' = z$, $w' = w$ evidently belongs to the orthogonal system, but is no rotation. It is a reflection on the lineoid $x = 0$. This reflection leaves all points in the planes of xy, xz, and xw fixed, while the planes of zw, yw, and yz are reflected on the axis of y.

A simple algebraic criterion serves to distinguish between the rotations and the conjugate transformations. The square of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

of any orthogonal transformation is, as is well known, equal to $+1$, and consequently the determinant itself is equal to either $+1$ or -1.

Those transformations whose determinant is $+1$ are rotations of the four-dimensional space about the origin.

If those transformations whose determinants are -1 be followed by a reflection on any lineoid through the origin, say by the reflection $x' = -x$, $y' = y$, $z' = z$, $w' = w$, the determinant of the resultant transformation is

$$\begin{vmatrix} -a_1 & b_1 & c_1 & d_1 \\ -a_2 & b_2 & c_2 & d_2 \\ -a_3 & b_3 & c_3 & d_3 \\ -a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

and is therefore $+1$. The resultant transformation is accordingly a rotation.

It appears, therefore, that all orthogonal transformations of determinant -1 are conjugate transformations.

Since the determinant of the combination of two linear transformations is the product of the determinants of the two component transformations, it follows that the combination of two rotations, or of two conjugate transformations, is a rotation, while the combination of a rotation with a conjugate transformation is a conjugate transformation.

The rotations accordingly form a *group*, while the conjugate transformations do not.

The conjugate transformations do not in general leave any point except the origin fixed. They may, however, leave a plane, and, in the particular case of reflections, a lineoid fixed.

The further treatment of this subject I reserve for another paper on groups of rotations in four-dimensional space, to which the present article is intended largely as a preface.

ANN ARBOR, Sept., 1889.

Sur les Equations aux Dérivées Partielles de la Physique Mathématique.

PAR H. POINCARÉ.

Quand on envisage les divers problèmes de Calcul Intégral qui se posent naturellement lorsqu'on veut approfondir les parties les plus différentes de la Physique, il est impossible de n'être pas frappé des analogies que tous ces problèmes présentent entre eux. Qu'il s'agisse de l'électricité statique ou dynamique, de la propagation de la chaleur, de l'optique, de l'élasticité, de l'hydrodynamique, on est toujours conduit à des équations différentielles de même famille et les conditions aux limites, quoique différentes, ne sont pas pourtant sans offrir quelques resemblances. Nous ne citerons ici que quelques exemples.

J'imagine d'abord que l'on se propose de trouver la température finale d'un corps solide conducteur, homogène et isotrope, lorsque les divers points de la surface de ce corps sont maintenus artificiellement à des températures données.

Ce problème traduit dans le langage analytique s'énonce comme il suit:

Trouver une fonction V qui dans une portion de l'espace satisfasse à l'équation de Laplace,

$$\Delta V = \frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0,$$

et qui prenne des valeurs données aux divers points de la surface qui limite cet espace.

C'est le *problème de Dirichlet.*

Supposons maintenant que l'on cherche quelle est la distribution de l'électricité statique à la surface d'un conducteur donné; nous retrouverons le même problème analytique.

Il s'agit de trouver une fonction V qui satisfasse à l'équation de Laplace dans tout l'espace extérieur au conducteur et qui se réduise à 0 à l'infini et à 1 à la surface au conducteur.

C'est un cas particulier du problème de Dirichlet, mais on connaît un moyen (par les fonctions de Green) de ramener le cas général à ce cas particulier.

Les deux problèmes, absolument différents au point de vue physique, sont identiques au point de vue analytique.

D'autres analogies, quoique moins completes, sont cependant évidentes.

Nous citerons d'abord le problème suivant; un liquide est contenu dans un vase qu'il remplit complètement; divers corps solides mobiles sont plongés dans ce liquide; on connaît les mouvements de ces corps et on suppose qu'il y a une fonction des vitesses; on demande quel est le mouvement du liquide.

C'est là le problème des sphères pulsantes de M. Bjerknes (imitation hydrodynamique des phénomènes électriques).

Au point de vue analytique, il s'agit de trouver une fonction V qui satisfasse à l'équation de Laplace à l'intérieur, d'un certain espace et telle que sur la surface qui limite cet espace la dérivée $\frac{dV}{dn}$ ait des valeurs données.

Je rappelle quel est le sens de cette notation $\frac{dV}{dn}$ dont il sera fait un fréquent usage dans la suite. Soit un élément de surface quelconque; et α, β, γ les trois cosinus directeurs de la normale à cet élément; nous posons:

$$\frac{dV}{dn}=\alpha\frac{dV}{dx}+\beta\frac{dV}{dy}+\gamma\frac{dV}{dz}.$$

Ainsi dans le problème hydrodynamique, nous retrouvons la même équation différentielle que dans les problèmes thermique et électrique; les conditions aux limites seules différent. Il en sera encore de même dans le problème de l'induction magnétique.

Supposons un ou plusieurs aimants permanents mis en présence d'un corps magnétique parfaitement doux M. Il s'agit de trouver une fonction V (le potentiel magnétique) qui satisfait à l'équation de Laplace dans toute la portion de l'espace qui n'est pas occupée par des aimants permanents et qui est assujettie en outre aux conditions suivantes. Aux divers points où il y a du magnétisme permanent, ΔV n'est pas nul, mais peut être regardé comme donné. La fonction V est continue dans tout l'espace; ses dérivées sont continues à l'intérieur du corps M et à l'extérieur de ce corps, mais elles sont discontinues à la surface du corps M. Dans le voisinage de cette surface, $\frac{dV}{dn}$ aura donc deux valeurs différentes selon qu'on se placera à l'intérieur

ou à l'extérieur du corps M; mais le rapport de ces deux valeurs sera une constante donnée.

Ici encore, nous retrouvons la même équation différentielle, avec des conditions aux limites analogues quoique différentes.

Voici maintenant des cas où l'équation différentielle est légèrement modifiée.

Supposons que l'on cherche la loi du refroidissement d'un corps solide isolé dans l'espace. Il s'agira de trouver une fonction V satisfaisant à l'équation

$$\frac{dV}{dt} = k\Delta V,$$

et qui de plus est donnée pour $t = 0$. Enfin à la surface du corps le rapport de V à $\frac{dV}{dn}$ est donnée.

Dans les problèmes d'optique, on a trois fonctions inconnues u, v, w et quatre équations :

$$\frac{d^2u}{dt^2} = k\Delta u, \quad \frac{d^2v}{dt^2} = k\Delta v, \quad \frac{d^2w}{dt^2} = k\Delta w, \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

Les conditions aux limites varient suivant les problèmes ; mais dans les questions de diffraction principalement elles ne sont pas sans analogie avec celles que nous avons rencontrées jusqu'ici.

Ce sont encore les mêmes équations, avec des conditions aux limites analogues, quoique différentes, que l'on rencontre dans le problème de la viscosité des liquides, traité d'après des idées de Navier. Les récents travaux de M. Couette ont rappelé l'attention sur cette question, qui était tombée dans un injuste oubli, malgré le beau chapitre que Kirchhoff y avait consacré dans sa Physique Mathématique.

La théorie de l'élasticité nous offre des équations plus compliquées, mais qui ne diffèrent pas beaucoup des précédentes.

On a encore trois fonctions inconnues u, v, w, auxquelles j'adjoindrai la fonction auxiliaire $\theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$ et trois équations dont la première s'écrit:

$$\Delta u + \lambda \frac{d\theta}{dx} = 0.$$

Je n'écris pas les conditions aux limites tout à fait analogues, *mutatis mutandis*, à celles des problèmes précédents et je passe immédiatement à une question très

importante en hydrodynamique et qui consiste à trouver les composantes de la vitesse en tous les points d'un liquide quand on connaît les composantes du tourbillon en tous les points de ce même liquide. Ce problème au point de vue analytique s'énonce comme il suit:

Connaissant trois fonctions α, β, γ, trouver trois fonctions inconnues u, v, w, qui satisfont à certaines conditions aux limites et de plus aux équations

$$\alpha = \frac{dw}{dy} - \frac{dv}{dz}, \quad \beta = \frac{du}{dz} - \frac{dw}{dx}, \quad \gamma = \frac{dv}{dx} - \frac{du}{dy}, \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \quad (1)$$

L'analogie avec les problèmes précédents ne paraît pas d'abord évidente, mais elle le devient si on observe que les trois premières équations (1) peuvent (en vertu de la quatrième) être remplacées par trois autres dont la première s'écrit

$$\Delta u = \frac{d\beta}{dz} - \frac{d\gamma}{dy},$$

et dont des autres peuvent s'écrire par symétrie.

Je pourrais montrer aussi, si je ne craignais de fatiguer l'attention par de trop nombreux exemples, que presque toutes les questions, encore mal étudiées, relatives à l'induction électrodynamique dans des conducteurs non linéaires se ramènent à des problèmes analogues aux précédents et surtout à la dernière question d'hydrodynamique que je viens de mentionner.

Cette revue rapide des diverses parties de la Physique Mathématique nous a convaincus que tous ces problèmes, malgré l'extrême variété des conditions aux limites et même des équations différentielles, ont, pour ainsi dire, un certain air de famille qu'il est impossible de méconnaître. On doit donc s'attendre à leur trouver un très grand-nombre de propriétés communes.

Malheureusement la première des propriétés communes à tous ces problèmes, c'est leur extrême difficulté. Non seulement on ne peut le plus souvent les résoudre complètement, mais ce n'est qu'au prix des plus grands efforts qu'on peut en démontrer rigoureusement la possibilité.

Cette démonstration est-elle nécessaire? La plupart des physiciens en feraient bon marché. L'expérience ne permettant pas de douter, par exemple, de la possibilité de l'équilibre électrique, on ne peut douter, non plus semble-t-il, de la possibilité des équations qui expriment cet équilibre. Nous ne saurions nous contenter de cette défaite; l'analyse doit pouvoir se suffire à elle-même et d'ailleurs un pareil raisonnement, n'il s'applique peut-être aux problèmes que

l'on rencontre directement en Physique, ne saurait s'appliquer de même à une foule de problèmes plus simples, qui se posent d'eux mêmes dès qu'on cherche à résoudre les premiers. En outre toute démonstration rigoureuse de la possibilité d'un problème en est toujours une solution ; dans le cas qui nous occupe, cette solution sera généralement grossière et tout à fait impropre au calcul numérique ; cependant elle nous enseignera toujours quelque chose.

Maintenant si cette démonstration est nécessaire, devons-nous pourtant nous astreindre à la même rigueur que dans une question d'analyse pure ? Ce serait dans beaucoup de cas un pédantisme bien inutile. Les équations différentielles auxquelles obéissent les phénomènes physiques n'ont été souvent établies que par des raisonnements peu rigoureux ; on ne les regarde que comme des approximation ; les résultats expérimentaux, auxquels il s'agit de comparer les conséquences de la théorie, sont eux-mêmes approximatifs. Dans ces conditions, la rigueur absolue est de peu de prix, et il semble souvent qu'il n'y a pas lieu de la rechercher si on doit la payer de trops d'efforts.

Mais alors comment reconnaîtra-t-on qu'on raisonnement dont la rigueur n'est pas absolue, n'est pas un simple paralogisme ? Quand aura-t-on le droit de dire que telle démonstration, insuffisante pour l'Analyse, est assez rigoureuse pour la Physique ? La limite est bien difficile à tracer. J'essayerai pourtant de le faire ; je m'efforcerai de marquer nettement cette frontière et d'expliquer pourquoi en deçà on est encore dans le domaine de la science, et au delà dans celui du paralogisme.

Néanmoins toutes les fois que je le pourrai, je viserai à la rigueur absolue et cela pour deux raisons ; en premier lieu, il est toujours dur pour un géomètre d'aborder un problème sans le résoudre complètement ; en second lieu, les équations que j'étudierai sont susceptibles, non seulement d'applications physiques, mais encore d'applications analytiques. C'est sur la possibilité du problème de Dirichlet que Riemann a fondé sa magnifique théorie des fonctions abélienne. Depuis d'autres géomètres ont fait d'importantes applications de ce même principe aux parties les plus fondamentales de l'Analyse pure. Est-il encore permis de se contenter d'une demi-rigueur ? Et qui nous dit que les autres problèmes de la Physique Mathématique ne seront pas un jour, comme l'a déjà été le plus simple d'entre eux appelés à jouer en Analyse un rôle considérable ?

§1.—*Problème de Dirichlet.*

Le Problème de Dirichlet énoncé plus haut est toujours possible; ce principe est connu sous le nom de principe de Dirichlet et la première démonstration est due à Riemann. Si une fonction V est assujettie à prendre des valeurs données aux divers points d'une certaine surface, limitant un certain volume où le fonction et ses dérivées sont continues, l'intégrale triple :

$$\int\int\int\left[\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right]dx\,dy\,dz$$

ne peut s'annuler; elle admet donc un certain minimum et il est aisé de vérifier que ce minimum correspond au cas où la fonction V satisfait à l'équation de Laplace. Cette démonstration, qui est à peu de chose près elle de Riemann n'est pas rigoureuse car elle est soumise à toutes les objections relatives à la continuité des fonctions définies par le calcul des variations.

Aussi un très grand nombre de géomètres se sont-ils préoccupés d'établir plus solidement ce principe; je citerai en première ligne les recherches de M. Schwarz (dans le programme de l'Ecole Polytechnique de Zurich 1869 et dans les Monatsberichte de l'Academie de Berlin 1870) bien qu'elles se rapportent plus particulièrement au cas de deux variables et ne puissent pas toujours s'étendre sans modification au cas qui nous occupe.

M. Neumann a donné de son côté une méthode générale qui permet de resoudre complètement le problème, si la surface où la fonction V prend des valeurs données est convexe. Il résout donc le problème de la distribution électrique dans le cas d'un conducteur convexe.

La méthode de M. Robin ne s'applique également qu'aux conducteurs convexes. Toutefois il y a un certain nombre de méthodes plus ou moins compliquées connues sous le nom de *méthodes alternantes* et qui permettent d'étendre les résultats au cas d'un conducteur de forme quelconque ou de plusieurs conducteurs isolés.

Le problème de Dirichlet se ramène à la recherche des fonctions de Green.

Soit à trouver une fonction V qui à l'intérieur d'une surface S satisfasse à l'équation de Laplace et sur cette surface prenne des valeurs données.

Supposons qu'on ait trouvé une fonction U qui soit finie, qui satisfasse à l'équation $\Delta U=0$, continue à l'intérieur de S sauf au point (a, b, c) intérieur à S où la fonction U sera infinie, de telle façon que la différence

$$U-\frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$

soit finie. Pour achever de définir la fonction U nous l'assujettirons à s'annuler en tous les points de S. On voit alors que la valeur de V au point a, b, c est égale à l'intégrale

$$\int \frac{V \frac{dV}{dn} d\omega}{4\pi}$$

étendue à tous les éléments $d\omega$ de S.

Donc quand on saura trouver les fonctions de Green, on saura résoudre le problème de Dirichlet.

D'autre part la recherche des fonctions de Green se ramène, à l'aide de la transformation par rayons vecteurs réciproques, aù problème de la distribution électrostatique à la surface d'un conducteur.

Ce problème de la distribution électrostatique n'est qu'un cas particulier du problème de Dirichlet, et cependant nous voyons que le cas général s'y ramène. La méthode alternante de Murphy permet d'ailleurs de ramener le problème de la distribution à la surface de plusieurs conducteurs isolés, au cas d'un conducteur unique. Nous ne nous occuperons donc plus désormais que du cas particulier où l'on cherche la distribution électrostatique à la surface d'un seul conducteur, puisque le cas général 1 y ramène.

Que devons-nous penser des méthodes proposées jusqu'ici? Ce sont à la fois des méthodes de démonstration destinées à établir la possibilité du problème et des méthodes de calcul destinées à le résoudre effectivement. Comme méthodes de démonstration, elles sont assez compliquées, mais elles se completent mutuellement de façon à s'appliquer à tous les cas et à satisfaire les juges plus sévères au sujet de la rigueur.

Comme méthodes de calcul, elles ne valent rien; car personne n'aura jamais l'idée de les appliquer; même les plus simples d'entre elles, celles de Neumann ou de Robin, conduisent à des calculs inextricables dès la seconde approximation. Tout ce qu'on peut espérer en tirer, sans un labeur par trop écrasant, ce sont des inégalités assez grossières auxquelles sera sommise la capacité du conducteur.

Tel est l'état actuel de la question; voyons maintenant quel est le but que je me suis proposé; ce qui eût été le plus intéressant, c'eût été de remplacer les méthodes de calcul actuel par d'autres moins défectueuses. Je n'ai pu le faire, je me suis borné à chercher une méthode de démonstration plus simple que celles qui ont été proposées jusqu'ici et directement applicable à tous les cas.

Je vais commencer par rappeler succinctement les principales propositions dont j'aurai à faire usage dans la suite.

1°. La fonction de Green, U, relativè à une sphère S et à un point P s'obtient de la façon suivante. Soit R le rayon de la sphère.

Prenons sur la droite OP une longueur $OQ = \frac{R^2}{OP}$; la fonction U sera le potentiel de deux masses l'une égale à 1 et placée au point P, l'autre égale à $-\sqrt{\frac{OQ}{OP}}$ et placée au point Q.

2°. La valeur de $\frac{dU}{dn}$ correspondant à un élément quelconque de la sphère S est en raison inverse du cube de la distance de cet élément au point P. Elle est égale à

$$\frac{R^2 - OP^2}{R} \frac{1}{MP^3},$$

M désignant le centre de gravité de l'élément considéré.

3°. Si l'on considère une sphère S et un point P intérieur à cette sphère, le potentiel d'une masse électrique égale à 1 et placée au point P sera égale à $\frac{1}{MP}$ au point M.

Imaginons ensuite que cette même masse électrique égale à 1 se répartisse sur la surface de la sphère S de telle façon que la densité sur un élément quelconque de cette sphère soit en raison inverse du cube de la distance de cet élément au point P. Je dis que le potentiel de cette masse ainsi distribuée que nous appellerons W sera égale à $\frac{1}{MP}$ en tout point M extérieur à la sphère et plus petit que $\frac{1}{MP}$ si le point M est intérieur à la sphère.

En effet considérons une fonction qui soit égale à la fonction de Green, U, définie plus haut à l'intérieur de la sphère et 0 à l'extérieur de la sphère. Cette fonction satisfait partout à l'équation de Laplace; elle est continue sauf au point P et sur la surface de la sphère. Au point P la fonction devient infinie et sa différence avec $\frac{1}{MP}$ reste finie; sur la surface de la sphère, la fonction elle-même reste continue, mais sa dérivée $\frac{d}{dn}$ subit un saut brusque égal à $\frac{R^2 - OP^2}{R.MP^3}$.

Nous devons en conclure que cette fonction est égale au potentiel de diverses masses électriques distribuées comme il suit:

1°. Une masse égale à 1 au point P.

2°. Une masse de densité $-\dfrac{R^2-OP^2}{4\pi.R.MP^3}$ aux divers points de la surface de S.

Cette seconde masse n'est autre chose que la masse définie plus haut et dont le potentiel était W, mais *changée de signe.*

On a donc:

$$\frac{1}{MP}-W=0$$

à l'extérieur de S et

$$\frac{1}{MP}-W=U>0$$

à l'intérieur de S. C. Q. F. D.

4°. Supposons que nous nous proposions de trouver une fonction V qui satisfasse à l'équation de Laplace à l'intérieur de la sphère S et qui aux différents points M de la surface de cette sphère prennent des valeurs données V^0.

La valeur de cette fonction V en un point P intérieur à la sphère sera l'intégrale

$$\int\frac{R^2-OP^2}{4\pi R}\,\frac{V^0}{MP^3}\,d\omega,$$

étendue à tous les éléments $d\omega$ de la sphère.

5°. Soient w et W la plus petite et la plus grande valeur que puisse prendre V^0; ce seront aussi, comme on le sait, la plus petite et la plus grande valeur que puisse prendre V.

Si w est positif, il sera de même de V^0 et de V.

D'ailleurs comme MP est toujours compris entre $R-OP$ et $R+OP$; on aura

$$V<\frac{R+OP}{(R-OP)^2}\int\frac{V^0d\omega}{4\pi R}$$

et

$$V>\frac{R-OP}{(R+OP)^2}\int\frac{V^0d\omega}{4\pi R}.$$

6°. J'arrive au théorème de Harnack dont je ferai un fréquent usage et qui peut s'énoncer comme il suit:

Si une série

$$V_1+V_2+\ldots.+V_n+\ldots.$$

est convergente en un point intérieur à sphère S, que tous ses termes soient positifs à l'intérieur de cette sphère et satisfassent tous à l'équation de Laplace, cette série sera uniformément convergente à l'intérieur de toute sphère intérieure elle-même à S.

Soit en effet V_i^0 la valeur de V_i en un point quelconque de la surface de S; d'après l'hypothèse V_i^0 sera essentiellement positif de sorte qu'on pourra écrire:

$$V_1 < \frac{R+OP}{(R-OP)^2}\int \frac{V_i^0 d\omega}{4\pi R}, \quad V_i > \frac{R-OP}{(R+OP)^2}\int \frac{V_i^0 d\omega}{4\pi R}.$$

Si maintenant nous appelons V_i' la valeur de V_i au point O, centre de la sphère, on aura

$$V_i' = \int \frac{V_i^0 d\omega}{4\pi R^2},$$

d'où:

$$V_i' \frac{R(R+OP)}{(R-OP)^2} > V_i > \frac{R(R-OP)}{(R+OP)^2} V_i',$$

ou si le point P est intérieur à une sphère s, concentrique à S et de rayon $r < R$,

$$\frac{R(R+r)}{(R-r)^2} V_i' > V_i > \frac{R(R-r)}{(R+r)^2} V_i'.$$

Si donc le point P est intérieur à s, chacun des termes positif de la série

$$V_1 + V_2 + \ldots + V_n + \ldots$$

sera plus petit que le terme de la série convergente

$$V_1' + V_2' + \ldots + V_n' + \ldots$$

multiplié par le facteur constant

$$\frac{R(R+r)}{(R-r)^2}.$$

Si donc la série converge au point O, elle convergera en tous les points intérieurs à S, et la convergence sera uniforme à l'intérieur de s.

Il suffit d'ailleurs que la série converge en un point P quelconque pour qu'elle converge au point O; on a en effet:

$$V_i' < V_i \frac{(R+OP)^2}{R(R-OP)}.$$

Ainsi si la série converge en un point quelconque intérieur à S, elle convergera encore en tous les autres points intérieurs à S, et la convergence sera uniforme dans toute sphère intérieure à S.

Ce n'est pas tout; non seulement la série:

$$V_1 + V_2 + \ldots + V_n + \ldots$$

converge uniformément à l'intérieur de s; mais il en est de même des séries

$$\frac{dV_1}{dx} + \frac{dV_2}{dx} + \ldots + \frac{dV_n}{dx} + \ldots,$$

$$\frac{d^2V_1}{dx^2} + \frac{d^2V_2}{dx^2} + \ldots + \frac{d^2V_n}{dx^2} + \ldots,$$

$$\frac{d^2V_1}{dxdy} + \frac{d^2V_2}{dxdy} + \ldots + \frac{d^2V_n}{dxdy} + \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

La démonstration est la même; et en effet la fonction V_i étant exprimée sous la forme d'une intégrale définie, une dérivée quelconque de V_i s'obtiendra sous la même forme par le procédé de la différentiation sous le signe $\int$; et on en déduira comme plus haut deux inégalités auxquelles cette dérivée devra satisfaire.

On en conclut qu'à l'intérieur de la sphère S, la somme de la série

$$V_1 + V_2 + \ldots + V_n + \ldots$$

est une fonction finie et continue ainsi que toutes ses dérivées et que cette fonction satisfait à l'équation de Laplace.

Le théorème démontré pour une sphère, s'étend aisément à une région R de l'espace.

Si dans la région R les fonctions

$$V_1,\ V_2,\ \ldots\ V_n,\ \ldots$$

sont positives et satisfont à l'équation de Laplace.

Si en un point de cette région la série

$$V_1 + V_2 + \ldots + V_n + \ldots$$

converge, elle convergera dans toute la région et la somme sera une fonction finie et continue qui satisfera à l'équation de Laplace.

Ces préliminaires posés, je puis aborder le problème que j'ai en vue. Il s'agit de démontrer la possibilité de l'équilibre électrostatique à la surface d'un conducteur isolé.

Je supposerai qu'en chaque point de la surface de ce conducteur il y a un plan tangent déterminé et deux rayons de courbure principaux déterminés. Ces

conditions restrictives ne sont pas toutes indispensables. Je préfère pourtant me les imposer d'abord et rechercher ensuite, par une courte discussion, quelles sont celles dont je puis me débarrasser.

On peut évidemment toujours trouver une sphère Σ telle que le conducteur soit contenu tout entier à l'intérieur de cette sphère.

On peut ensuite trouver un système de sphères en nombre infini

$$S_1, S_2, \ldots, S_n, \ldots$$

jouissant des deux propriétés suivantes: 1°. Chacune de ces sphères sera tout entière extérieure au conducteur. 2°. Tout point de l'espace extérieur au conducteur appartiens aù moins à une des sphères du système.

Cette proposition paraîtra presque évidente à qui voudra se donner la peine d'y réfléchir.

Nous croyons néanmoins devoir la démontrer en quelques mots.

Il s'agit de démontrer que l'on peut trouver à l'extérieur du conducteur une infinité de points

$$C_1, C_2, \ldots, C_n, \ldots$$

tels que pour tout point M extérieur au conducteur, il y ait un point C_i plus rapproché du point M que de la surface du conducteur. Et en effet s'il en est ainsi et que du point C_i comme centre on décrive une sphère S_i ayant pour rayon la plus courte distance de C_i à la surface du conducteur, la sphère S_i sera tout entière extérieure au conducteur et le point M sera intérieur à la sphère S_i.

Cherchons donc à établir l'existence des points C_i.

Considérons d'abord une région finie R située à une distance finie de la surface du conducteur et soit δ la plus courte distance de la région R à cette surface. Cela posé considérons le triple système de plans:

$$x = \frac{m_1\delta}{\sqrt{3}}, \quad y = \frac{m_2\delta}{\sqrt{3}}, \quad z = \frac{m_3\delta}{\sqrt{3}}$$

parallèles aux trois plans de coordonnées. Les trois quantités m_1, m_2, m_3 sont des nombres entiers positifs ou négatifs. Ces plans partageront l'espace en une infinité de cubes égaux dont la diagonale est égale à δ.

Ceux de ces cubes qui appartiendront en totalité ou en partie à la région R seront en nombre fini. Soient

$$K_1, K_2, \ldots, K_n$$

ces cubes. L'intérieur de chacun d'eux j'envisagerai un point appartenant à la région R; j'obtiendrai ainsi n points

$$C_1,\ C_2,\ \ldots\ldots,\ C_n,$$

le point C_i appartenant à la fois au cube K_i et à la région R.

Si nous considérons maintenant un point M quelconque de la région R, ce point appartiendra à l'un de nos cubes, par exemple au cube K_i. La distance du point M au point C_i sera plus petite que δ diagonale du cube; la distance du point C_i à la surface du conducteur est elle-même plus grande que δ, puisque C_i appartient à R.

Donc tout point de R est plus rapproché de l'un des n points $C_1,\ C_2,\ \ldots.\ C_n$ que celui-ci ne l'est du conducteur.

Je partage maintenant l'espace extérieur au conducteur en une infinité de régions

$$\ldots.\ R_{-n},\ R_{-n+1},\ \ldots.,\ R_{-2},\ R_{-1},\ R_0,\ R_1,\ \ldots.,\ R_n,\ \ldots.$$

Chacune de ces régions est ainsi représentée par la lettre R affectée d'un indice qui peut varier depuis $-\infty$ jusqu'à $+\infty$. La région R_i se compose des points dont la plus courte distance à la surface du conducteur est comprise entre Q^i et Q^{i+1}. Dans chacune de ces régions je définirai les points $C_1,\ C_2,\ \ldots,\ C_n$ comme je l'ai dit plus haut. J'obtiendrai ainsi une infinité de points C. Il est clair alors que tout point extérieur au conducteur sera toujours plus près de l'un de ces points que celui-ci ne l'est lui-même de la surface du conducteur.

C. Q. F. D.

L'existence des sphères S_i est donc établie.

Il est clair qu'on pourrait construire ces sphères S_i d'une infinité d'autres manières et que celle que je viens d'exposer en détail dans le seul but de fixer les idées n'a été choisie d'une façon tout à fait arbitraire. La seule chose essentielle c'est que les sphères S_i, quoique en nombre infini, puissent être rangées en une série linéaire:

$$S_1,\ S_2,\ \ldots.,\ S_n,\ \ldots.,$$

où chacune d'elles ait un indice entier positif parfaitement déterminé, c'est-à-dire pour parler le langage de M. Cantor, que l'ensemble des sphères S_i soit de la 1ère puissance.

L'ordre dans lequel ces sphères seront rangées dans la série linéaire

$$S_1,\ S_2,\ \ldots.,\ S_n,\ \ldots.$$

est d'ailleurs indifférent.

Il s'agit maintenant d'établir qu'il existe une fonction V finie et continue ainsi que toutes ses dérivées à l'extérieur du conducteur et satisfaisant à l'extérieur du conducteur à l'équation de Laplace; et de plus que cette fonction tend vers 0 quand le point (x, y, z) s'éloigne indéfiniment et vers 1 quand le point (x, y, z) se rapproche indéfiniment de la surface du conducteur.

Soit O le centre et R le rayon de la sphère Σ; imaginons une certaine quantité d'électricité positive distribuée uniformément à la surface de cette sphère avec une densité $\frac{1}{4\pi R}$. J'appelle V_0 le potentiel dû à cette électricité. Nous aurons

$$V_0 = 1$$

à l'intérieur de la sphère Σ et en particulier à l'intérieur du conducteur et

$$V_0 = \frac{R}{OM}$$

lorsque le point $M(x, y, z)$ est extérieur à Σ. On a dans tous les cas

$$0 < V_0 \leqq 1$$

et V_0 tend vers 0 quand le point M s'éloigne indéfiniment.

Nous allons faire maintenant une série d'opérations que je vais définir.

Nous avons vu plus haut que si une masse électrique P se trouve à l'intérieur d'une sphère S, on peut la remplacer par une masse égale distribuée à la surface de la sphère de façon que la densité en chaque point de cette surface soit en rayon inverse du cube de la distance de ce point au point P. Le potentiel par rapport à un point extérieur à S n'est pas changé, le potentiel par rapport à un point intérieur est diminué.

On peut appeler cette couche électrique répandue à la surface de S la *couche équivalente* à la masse unique P.

Cela posé, supposons qu'on remplace toutes les masses électriques qui peuvent exister à l'intérieur d'une sphère S_i par la couche équivalente répandue à la surface de cette sphère. Le potentiel en un point extérieur à S_i ne changera pas, le potentiel en un point intérieur à S_i diminuera. Cette opération pourra s'appeler: "balayer la sphère S_i."

Nous partons de la masse électrique répandue sur Σ et dont le potentiel est V_0. Chacun des points de Σ appartenant à l'une des sphères S_i, quelques-unes de ces sphères contiendront de l'électricité; soit S_1 une de celles qui en contien-

nent; "balayons" cette sphère; balayons ensuite la sphère S_2 si cette sphère contient de l'électricité et ainsi de suite.

Il faut diriger les opérations de façon que chaque sphère soit balayée une infinité de fois. On peut par exemple balayer les sphères dans l'ordre suivant:

$$S_1S_2S_1S_2S_3S_1S_2S_3S_4S_1S_2S_3S_4S_5S_1S_2S_3S_4S_5S_6 \ldots .$$

Il est aisé de constater que de cette façon chaque sphère est balayée une infinité de fois.

Soit ensuite V_1 ce que devient le potentiel V_0 après la première opération, V_2 ce que devient V_1 après la seconde opération; soit enfin V_n ce qui devient le potentiel après n opérations.

Supposons que la n^e opération consiste à balayer le sphère S_k. On aura:

$$V_n = V_{n-1} \quad \text{à l'extérieur de } S_k,$$
$$V_n < V_{n-1} \quad \text{à l'intérieur de } S_k.$$

On aura donc dans tous les cas:

$$V_n \leqq V_{n-1}.$$

Cette inégalité montre qu'en un point quelconque de l'espace V_n est toujours décroissant (ou du moins toujours non-croissant) quand on fait croitre l'indice n.

Il importe de remarquer qu'il n'y a aucun moment et en aucun point de masse électrique négative. Au début nous avons sur la sphère Σ une couche électrique uniforme et positive. Aucune des opérations subséquentes ne peut introduire de masses négatives. En effet le balayage d'une sphère quelconque consiste à remplacer les masses électriques *positives* situées à l'intérieur de cette sphère par des couches équivalentes *positives* répandues à la surface de cette sphère.

On a donc $$V_n > 0.$$

Ainsi en un point quelconque de l'espace V_n est toujours positif et décroissant. Donc quand n croît indéfiniment, V_n tend vers une limite finie et déterminée que j'appelle V. J'ai donc démontré l'existence d'une fonction V définie en tous les points de l'espace. Il reste à étudier les propriétés de cette fonction.

Considérons une quelconque des sphères S_i. Par hypothèse cette sphère sera balayée une infinité de fois. Supposons qu'elle le soit à la α_1^o opération, à la α_2^o, ... à la α_n^o, ... Après chacune de ces opérations, il n'y aura plus d'électricité à l'intérieur de S_i de sorte qu'on aura:

$$\Delta V_\alpha = 0 \qquad (\alpha_i = \alpha_1, \alpha_2, \alpha_3, \ldots . \alpha_n, \ldots .).$$

Quand α_n croît indéfiniment, V_{α_n} tend vers V de sorte que la série:

$$V_{\alpha_1} + (V_{\alpha_2} - V_{\alpha_1}) + (V_{\alpha_3} - V_{\alpha_2}) + \ldots + (V_{\alpha_n} - V_{\alpha_{n-1}}) + \ldots$$

converge et a pour somme V. Chacun de ces termes satisfait à l'équation de Laplace; de plus chacun d'eux est négatif (sauf le premier) car on a:

$$V_{\alpha_n} < V_{\alpha_{n-1}} \text{ si comme on le suppose } \alpha_n > \alpha_{n-1}.$$

Donc en vertu du théorème de Harnack la convergence est uniforme et les séries déduites de la précédente par différentiation convergent aussi uniformément. Donc à l'intérieur de S_i la fonction V est continue aussi que toutes ses dérivées et satisfait à l'équation de Laplace.

Mais par hypothèse tout point de l'espace éxtérieur au conducteur appartient au moins à l'une des sphères S_i.

Donc en tout point extérieur au conducteur, V et ses dérivées sont continues et l'on a:

$$\Delta V = 0.$$

Je dis que V tend vers 0 quand le point M s'éloigne indéfiniment. En effet on a:

$$V_0 > V > 0.$$

Or V_0 tend vers 0 quand M s'éloigne indéfiniment, donc il en est de même de V.

Il reste à démontrer que V tend vers 1 quand le point M se rapproche indéfiniment de la surface du conducteur.

Soit donc M_0 un point de la surface du conducteur et suppose que le point M se rapproche indéfiniment de M_0. Par hypothèse il y a au point M_0 un plan tangent déterminé et deux rayons de courbure principaux déterminés. On peut donc construire une sphère σ tangente à la surface du conducteur en M_0 et dont le rayon r est assez petit pour que cette sphère soit tout entière contenue à l'intérieur du conducteur.

Soit C le centre de cette sphère. La fonction

$$\frac{M_0C}{MC} = \frac{r}{MC},$$

regardée comme fonction des coordonnées x, y, z du point M satisfait à l'équation de Laplace et si réduit à 1 à la surface de la sphère σ.

D'autre part la fonction V_n est un potentiel dû à des masses électriques qui sont toutes positives. Il en résulte que cette fonction V_n peut avoir des maxima, mais ne peut avoir de minimum.

De plus V_n est égal à 1 en tous les points intérieurs au conducteur et en particulier sur toute la surface de σ. En effet cela est vrai de V_0, mais, en balayant l'une des sphères S_i, on ne change pas le potentiel à l'extérieur de cette sphère, ni par conséquent à l'intérieur du conducteur qui est tout entier extérieur, par hypothèse, à toutes les sphères S_i. On a donc à l'intérieur du conducteur

$$1 = V_0 = V_1 = \ldots\ldots = V_n = \ldots\ldots$$

La différence

$$U = V_n - \frac{r}{MC}$$

est un potentiel dû aux masses électriques qui engendrent V_n et qui toutes sont positives et extérieures à σ, et à une masse $-r$ concentrée au point C et par conséquent intérieure à σ. Toutes celles de ces masses qui sont extérieures à σ sont positives. Donc à l'extérieur de σ, la fonction U ne peut avoir que des maxima et pas de minima. A la surface de σ, U est nul, car

$$V_n = \frac{r}{MC} = 1.$$

A l'infini, U est encore nul, car

$$V_n = \frac{r}{MC} = 0.$$

Donc à l'extérieur de σ, U ne peut être que positif, en sorte que l'on a:

$$V_n > \frac{r}{MC}.$$

Nous avons donc la double inégalité

$$1 > V_n > \frac{r}{MC},$$

qui à la limite devient:

$$1 \geqq V \geqq \frac{r}{MC} \geqq \frac{r}{r + MM_0}.$$

Il est donc clair que quand la distance MM_0 tend vers 0, V tend vers 1.

C. Q. F. D.

La fonction V satisfait donc bien aux conditions que nous nous etions imposées et le *principe de Dirichlet est établi.*

Voyons maintenant si on peut se débarrasser des conditions restrictives que nous nous sommes imposées, à savoir que le plan tangent et les rayons de courbure principaux sont déterminés. Nous ne nous sommes servis de ces condi-

tions que pour établir l'existence de la sphère σ, tangente au conducteur et toute entière intérieure à ce conducteur. Cette sphère σ n'existera évidemment que si le plan tangent est déterminé; mais elle pourra exister encore en un point singulier où les rayons de courbure ne varieraient pas suivant les lois habituelles; nous n'aurions alors rien à changer à notre démonstration.

Nous n'avons donc qu'à examiner les points singuliers pour lesquels la sphère σ n'existerait pas. S'il y a sur la surface du conducteur de pareils points singuliers, nous ne pouvons pas encore affirmer que la fonction V tende vers 1 quand le point M se rapproche de l'un de ces points singuliers; remarquons toutefois que l'existence de ces points singuliers n'empêche pas V de tendre vers 1 quand M se rapproche d'un point non singulier de la surface.

Mais on peut aller plus loin; supposons que le principe de Dirichlet soit démontré pour un certain conducteur C'. Supposons ensuite que le conducteur donné C présente un point singulier M_0, et que l'on puisse construire un conducteur C'', semblable à C', dont la surface passe par M_0 et qui soit tout entier intérieur à C. Je dis que quand le point M se rapprochera de M_0, V tendra vers 1.

En effet le principe de Dirichlet établi pour C' l'est aussi pour C''; il existe donc une fonction V'' satisfaisant à l'équation de Laplace à l'extérieur de C'' et se réduisant à 0 à l'infini et à 1 à la surface de C''. Nous allons faire jouer alors au conducteur C'' le même rôle qu'à la sphère σ dans le cas précédemment examiné. On verrait comme plus haut que:

$$1 > V_n > V''$$

et par conséquent que: $$1 > V > V''.$$

Or quand M tend vers M_0: $$\lim V'' = 1.$$

Donc $$\lim V = 1.$$ C. Q. F. D.

Parmi ces points singuliers les plus intéressants sont les points coniques que nous allons étudier de plus près.

Je suppose qu'une partie de la surface du conducteur C soit une portion de cône de révolution. Soit M_0 le sommet de ce cône. Je dis que V tendra vers 1 quand le point M se rapprochera indéfiniment de M_0.

Mais pour cela il est nécessaire de démontrer le lemme suivant:

Soit S une sphère fixe, C un cercle fixe sur cette sphère, P un point fixe

en dehors de la sphère. Soit maintenant E un ellipsoïde de révolution variable assujetti à restes tangent à la sphère S tout le long du cercle C. Soit V une fonction satisfaisant à l'équation $\Delta V = 0$ à l'extérieur de E et se réduisant à 0 à l'infini et à 1 à la surface de E. Soit V_0 la valeur de V au point P.

Je dis que V_0 tendra vers 1 quand le grand axe de E croîtra indéfiniment.

Désignons en effet par ρ et $\sqrt{\rho^2 - b^2}$ les axes de l'ellipsoïde E, par $\rho + h$ et $\sqrt{(\rho + h)^2 - b^2}$ les axes de l'ellipsoïde E', homofocal à E et passant par P. On aura :

$$V_0 = \frac{L\dfrac{\rho + h - b}{\rho + h + b}}{L\dfrac{\rho - b}{\rho + b}} = \frac{L(\rho + b) + L\left(1 + \dfrac{h}{\rho + b}\right) - L(\rho + h - b)}{L(\rho + b) - L(\rho - b)}.$$

A la limite les ellipsoïdes E et E' se réduisent à deux paraboloïdes homofocaux P_1 et P_1' faciles à construire.

Alors h tend vers une limite finie qui n'est autre que la distance des sommets des deux paraboloïdes ; $\rho - b$ tend vers une limite finie qui est le demiparamètre de P_1. On voit aussi que $L(\rho - b)$ et $L(\rho + h - b)$ tendent vers des limites finies, que $L\left(1 + \dfrac{h}{\rho + b}\right)$, tend vers 0 et que $L(\rho + b)$ croît indéfiniment. Par conséquent V_0 tend vers 1. C. Q. F. D.

Revenons maintenant à notre conducteur C dont une partie de la surface appartiendra à un cône de révolution de sommet M_0. Par le point M_0, et en dehors du cône faisons passer une droite M_0P et supposons que le point M se rapproche de M_0 en suivant cette droite ; je dis que V tendra vers 1.

En effet du point M je mène une normale au cône et par le pied de cette normale je fais passer un parallèle du cône de révolution. Je construis ensuite un ellipsoïde de révolution E qui soit tangent au cône tout le long de ce parallèle. Je ferai varier cet ellipsoïde de telle façon qu'il reste constamment tout entier à l'intérieur du conducteur C et son grand axe reste fini quand le point M se rapproche indéfiniment de M_0. Cela est manifestement toujours possible.

Je construis ensuite un potentiel W qui se réduise à 1 à la surface de E. Soit W_0 la valeur de W au point M, on aura :

$$1 > V > W_0.$$

Construisons maintenant une figure homothétique de la précédente en prenant pour centre d'homothétie le point M_0, le cône de révolution sera son propre

homothétique; l'homothétique de l'ellipsoïde E sera un ellipsoïde E' tangent au cône tout le long d'un parallèle; l'homothétique du point M sera un point M' de la droite M_0P. Nous choisirons le rapport d'homothétie de façon que M' soit fixe; alors l'ellipsoïde de révolution E sera tangent au cône tout le long d'un parallèle fixe.

Soit maintenant un potentiel W' qui se réduise à 1 à la surface de E'. Soit W'_0 la valeur de W' au point M'; on aura:

$$W_0 = W'_0.$$

Lorsque le point M se rapprochera de M_0, le rapport d'homothétie et par conséquent le grand axe de E' croîtront indéfiniment. Donc d'après le lemme W'_0 tendra vers 1. Donc W_0 et par conséquent V tendront aussi vers 1.

C. Q. F. D.

Cela posé considérons maintenant un conducteur C quelconque, et un point singulier M_0 de ce conducteur; je supposerai qu'en ce point M_0 le plan tangent est déterminé, ou que ce point M_0 est un point conique ordinaire. Je pourrai alors construire un conducteur C'' formé par exemple d'une sphère et d'un cône de révolution circonscrit, ayant pour sommet M_0; je pourrai choisir l'angle du sommet du cône et le rayon de la sphère assez petits pour que C'' soit tout entier intérieur à C.

Le principe de Dirichlet est démontré pour C''; C'' est intérieur à C; nous devons en conclure, comme nous l'avons vu plus haut que V tend vers 1, quand M tend vers M_0.

Nous pouvons résumer cette discussion en disant que *le principe de Dirichlet est établi pour tout conducteur dont la surface est telle que le plan tangent en chaque point est déterminé sauf en un nombre limite de points coniques ordinaires.*

Comme méthode de démonstration, celle que je viens d'exposer ne laisse rien à désirer, comme méthode de calcul; elle ne vaut pas mieux que celles qu'on a proposées jusqu'ici et même si le conducteur est convexe, elle est inférieure à celles de Neumann et de Robin. Mais même à ce point de vue, je ne regrette pas de l'avoir publiée; en effet, ainsi que nous l'avons vu, il ne faut guère compter que sur la première approximation si l'on veut pousser plus loin, même avec le procédé de Neumann, on serait conduit à des calculs trop rebutants. Chaque méthode donnera donc seulement quelques inégalités, il n'est donc pas inutile de multiplier les méthodes; dans chaque cas particulier, un analyste habile saura choisir celle qui convient le mieux, ou mieux encore les combiner toutes d'une manière convenable.

A ce point de vue, la méthode que j'expose ici offrira à cette analyste habile d'assez grandes ressources à cause de son élasticité (si j'ose m'exprimer ainsi). Le choix des sphères S_1, S_2, ... S_n reste arbitraire dans une très large mesure; d'ailleurs on peut sans rien changer à la méthode remplacer la sphère Σ par d'autres surfaces, ou même prendre pour V_0 une fonction potentielle quelconque, pourvu qu'elle se réduise à 1 à l'intérieur du conducteur et que les masses électriques qui l'engendrent soient toutes positives.

On pourra encore remplacer les sphères S_i par d'autres surfaces, pourvu que l'on sache construire sur cette surface la couche équivalente à une masse électrique donnée intérieure à la surface. On conçoit qu'on pourra profiter de toutes ces facilités pour adapter le mieux possible la méthode à chaque cas particulier.

Parmi ces perfectionnements dont la méthode est susceptible, il en est un qui me paraît fort important. Nous avons construit les sphères S_i de façon que tout point extérieur au conducteur soit intérieur au moins à l'une de ces sphères.

Imaginons qu'on construisa seulement assez de sphères S_i tout entières extérieures au conducteur, pour que tout point extérieur au conducteur et *intérieur* à Σ soit intérieur au moins à l'une de ces sphères. Les sphères S_i dont le rayon doit être fini dès qu'on est à une distance finie du conducteur, empéiteront d'ailleurs sur la région extérieure à Σ.

Dans ces conditions tout point de l'espace est ou bien intérieur au conducteur C, ou intérieur à l'une des sphères S_i, ou extérieur à Σ.

Il ne peut être à la fois intérieur à C et intérieur à S_i, ou bien intérieur à C et extérieur à Σ; mais il peut être à la fois intérieur à deux ou plusieurs des sphères S_i, ou intérieur à l'une ou plusieurs de ces sphères et extérieur à Σ. Nous avons vu plus haut comment on pouvait remplacer une masse électrique P intérieure d'une sphère Σ, c'est-à-dire par une couche électrique répandue à la surface de la sphère et dont le potentiel soit égal à celui de la masse P à l'extérieur de Σ et plus petit à l'intérieur.

Je dis maintenant qu'on peut également remplacer une masse électrique Q *extérieure* à la sphère Σ par une couche équivalente, c'est-à-dire par une couche électrique répandue à la surface de Σ, et dont le potentiel soit égal à celui de la masse Q à *l'intérieur* de Σ et plus petit à l'extérieur.

En effet soit O le centre de Σ et R son rayon; soient P et Q deux points situés sur un même rayon vecteur OP et tels que:

$$OP \cdot OQ = R^2.$$

Supposons que P soit intérieur à Σ; Q sera extérieur à Σ.

Considérons deux masses l'une égale à 1 et placée en P, l'autre égale à $-\sqrt{\frac{OQ}{OP}}$ et placée en Q. Nous avons vu que le potentiel dû à ces deux masses était une fonction U, nulle à la surface de Σ, positive à l'intérieur de Σ et négative à l'extérieur.

Soit W le potentiel de la couche équivalente à la masse P répandue à la surface de Σ. D'après ce que nous avons vu, nous aurons:

$$\text{à l'extérieur de } \Sigma: \quad \frac{1}{MP} - W = 0 \qquad \frac{1}{MP} < \sqrt{\frac{OQ}{OP}}\,\frac{1}{MQ}$$

et

$$\text{à l'intérieur de } \Sigma: \quad \frac{1}{MP} - W = \frac{1}{MP} - \sqrt{\frac{OQ}{OP}}\,\frac{1}{MQ} > 0.$$

La lettre M désigne toujours le point de coordonnées courantes x, y, z.

Ces égalités montre que l'on a:

$$\text{à l'extérieur de } \Sigma: \quad W < \sqrt{\frac{OQ}{OP}}\,\frac{1}{MQ},$$

$$\text{à l'intérieur de } \Sigma: \quad W = \sqrt{\frac{OQ}{OP}}\,\frac{1}{MQ}.$$

Par conséquent la couche équivalente à la masse intérieure 1 située en P est aussi équivalente à la masse extérieure $\sqrt{\frac{OQ}{OP}}$ située en Q.

Il importe de remarquer une différence essentielle entre les deux cas; nous savons que la masse totale de la couche équivalente à la masse 1 située en P est égale à 1 et par conséquent plus petite que $\sqrt{\frac{OQ}{OP}}$, c'est-à-dire que la masse extérieure située en Q.

Ainsi la couche équivalente à une masse *intérieure* a une masse totale *égale* à cette masse; la couche équivalente à une masse *extérieure* a une masse totale *plus petite* que cette masse.

A côté opérations dont il a été question plus haut et qui consistent à "balayer" l'intérieur des sphères S_i, nous pourrons introduire une opération nouvelle que nous pourrons appeler le balayage de l'extérieur de Σ.

Cette opération consistera à remplacer toutes les masses extérieures à Σ par la couche équivalente répandue à la surface de cette sphère.

Cela posé on fera une série de balayages successifs en partant de la fonction V_0 et en ayant soin de diriger les opérations de telle façon que l'intérieur de chacune des sphères S_i, ainsi que l'extérieur de Σ, soient balayés une infinité de fois.

Les raisonnements que nous avons faits plus haut sont encore applicables et on obtiendra une série de fonctions

$$V_0,\ V_1,\ \ldots\ldots,\ V_n,\ \ldots\ldots$$

qui convergeront de la fonction cherchée V.

Seulement l'approximation sera plus rapide, parce que chaque balayage de l'extérieur de Σ, peut remplacer le balayage d'une infinité de sphères S_i qui devraient, dans la méthode primitive, remplir l'espace infini extérieur à Σ.

De plus, nous devons observer que dans la méthode primitive, chaque balayage laissait subsister la même quantité totale d'électricité dans l'espace; au contraire, dans la méthode nouvelle, cette quantité totale diminue après chaque balayage de l'extérieur de Σ.

La méthode nouvelle se prête par conséquent mieux que la première au calcul des capacités.

Nous avons vu que le problème général de Dirichlet se ramène au cas que nous venons de traiter. Nous pourrions donc nous dispenser d'étudier directement ce cas général. Cependant je vais montrer que la méthode exposée ci-dessus y est directement applicable.

Soit donc C une surface partageant l'espace en deux régions l'une extérieure à la surface, l'autre intérieure. Soit U une fonction quelconque, bien déterminée en tous les points de C.

Nous nous proposons de trouver une fonction V:

1°. Qui est finie et continue ainsi que ses dérivées tant à l'extérieur de C qu'à l'intérieur, mais qui peut devenir discontinue sur la surface C elle-même.

2°. Qui satisfasse à l'équation de Laplace tant à l'extérieur de C qu'à l'intérieur, mais qui peut cesser d'y satisfaire sur la surface C elle-même.

3°. Qui tende vers O quand le point M de coordonnées courantes x, y, z s'éloigne indéfiniment.

4°. Qui tende vers U quand le point M se rapproche indéfiniment d'un point de C, soit par l'intérieur, soit par l'extérieur.

1^er^ cas. Supposons qu'on puisse trouver une fonction V_0 finie et continue dans tout l'espace ainsi que ses dérivées des deux 1^ers^ ordres; qui soit égale à 0 à

l'infini et à U à la surface de C et qui soit telle que ΔV_0 soit constamment négatif.

Une pareille fonction pourra être regardée comme un potentiel dû à de l'électricité répandue dans tout l'espace et dont la densité $-\frac{\Delta V_0}{4\pi}$ sera partont positive.

Nous n'avons donc encore ici que des masses électriques positives.

Construirons maintenant une infinité de sphères $S_1, S_2, \ldots, S_n$, de telle façon que chacune de ces sphères soit tout entière extérieure à la surface C et que tout point de l'espace, extérieur à C, soit intérieur au moins à l'une des sphères S_i.

Balayons ensuite, comme il a été dit plus haut, ces diverses sphères S_i en dirigeant les opérations de telle sorte que chacune d'elles soit balayée une infinité de fois.

Soit V_n ce que devient V_0 après la n^e opération; on aura encore, puisque toutes les masses électriques sont positives:

$$V_{n+1} < V_n \quad \text{et} \quad V_n > 0$$

ce qui montre que quand n croît indéfiniment V_n tend vers une limite finie et déterminée que j'appelle V. Pour un point intérieur à C on a $V_n = V_0$.

On verrait, comme plus haut, qu'à l'intérieur de chacune des sphères S_i, et par conséquent pour tout point extérieur à C, la fonction V est finie et continue ainsi que ses dérivées et satisfait à l'équation de Laplace.

On verrait également que V tend vers 0 quand le point M s'éloigne indéfiniment. Il me reste à montrer que V tend vers U quand le point M se rapproche de la surface C.

Nous supposerons pour fixer les idées que le point M se rapproche indéfiniment d'un point M_0 de la surface C en restant extérieur à cette surface.

Nous construirons une sphère σ tangente à la surface C en M_0 et de rayon assez petit pour être tout entière intérieure à C.

Le principe de Dirichlet étant démontré pour une sphère, nous pourrons construire une fonction W qui à l'extérieur de la sphère satisfasse à l'équation $\Delta W = 0$, qui se reduise à O à l'infini, et à V_0 à la surface de la sphère.

Quand le point M tendra vers M_0, la fonction W tendra vers V_0 et par conséquent vers U.

Comparons les fonctions V_n et W.

A la surface de σ, on a:

$$V_n = V_0 = W.$$

A l'infini, on a:

$$V_n = 0 = W.$$

Maintenant V_n et W sont deux potentiels; le premier est engendré par des masses électriques toutes positives et dont quelques unes sont extérieures à σ; le second par des masses qui sont toutes intérieures à σ.

Donc à l'extérieur de σ la différence $V_n - W$ peut avoir des maxima, mais pas de minima; et comme elle est nulle à la surface de σ elle sera toujours positive. On a donc:

$$V_0 > V_n > W \text{ et par conséquent à la limite } V_0 > V > W.$$

Quand M tend vers M_0, V_0 et W tendent tous deux vers U. Donc V tend aussi vers U. C. Q. F. D.

Le principe de Dirichlet est ainsi établi pour la région extérieure à C; on le démontrerait absolument de la même manière pour la région intérieure.

2e cas. Supposons maintenant qu'on puisse trouver une fonction V_0 qui soit finie et continue ainsi que ses dérivées des deux premiers ordres, qui se réduise à O à l'infini et à U à la surface de C. (Nous ne supposons donc plus que ΔV_0 est toujours négatif.) On pourra trouver une fonction V_0 satisfaisant à ces conditions toutes les fois que *la fonction U sera elle-même finie et continue ainsi que ses dérivées des deux premiers ordres.*

La fonction V_0 pourra être regardée comme un potentiel engendré par des masses électriques répandues dans tout l'espace avec une densité $-\frac{\Delta V_0}{4\pi}$. Seulement comme ΔV_0 n'est pas toujours négatif, les masses ne seront pas toutes positives. Nous pourrons alors écrire:

$$V_0 = V_0' - V_0'',$$

V_0' étant le potentiel dû aux masses positives seulement, et V_0'' le potentiel dû seulement aux masses négatives changées de signe.

Soient U' et U'' les valeurs de V_0' et V_0'' à la surface de C, on aura:

$$U = U' - U''.$$

V_0' et V_0'' n'étant engendrés que par des masses positives, on pourra, comme nous venons de le voir, construire des fonctions V' et V'' qui satisferont à l'équation

de Laplace à l'extérieur de C et qui se réduiront respectivement à U' et à U'' à la surface de C.

La différence $$V = V' - V''$$

satisfera à l'équation $\Delta V = 0$ et se réduira à U à la surface de C.

Le principe de Dirichlet est donc encore ici établi.

3e cas. Il nous reste à examiner le cas où la fonction U n'est plus continue ainsi que ses dérivées des deux premiers ordres.

Bien des méthodes s'offrent à nous pour généraliser les résultats obtenus dans les deux premiers cas.

Je ferai d'abord observer que si la fonction U est elle-même continue et si ses dérivées du 1er ordre ne présentent de discontinuités que le long de certaines courbes analytiques tracées sur la surface C, les démonstrations dont j'ai fait usage dans les deux premiers cas peuvent se répéter sans qu'on ait rien à y changer.

Passons au cas général; nous pourrons trouver deux séries indéfinies de fonctions $$U_1,\ U_2,\ \ldots,\ U_n,\ \ldots;\ U_1',\ U_2',\ \ldots,\ U_n',$$

qui à la surface de C jouissent des propriétés suivantes:

1°. Elles sont finies et continues ainsi que leurs dérivées des deux premiers ordres.

2°. On a: $$U_{n+1} > U_n,\quad U_{n+1}' < U_n',\quad U_n' > U_n.$$

3°. On a: $$\lim U_n = \lim U_n' = U \quad \text{pour } n \text{ infini.}$$

(Cela n'aura lieu que pour les points de C dans le voisinage desquels U est continue.)

Nous pourrons alors construire deux séries de fonctions

$$V_1,\ V_2,\ \ldots,\ V_n,\ \ldots;\ V_1',\ V_2',\ \ldots,\ V_n'$$

telles que $$\Delta V_n = 0 \quad \Delta V_n' = 0$$

à l'extérieur de C et $$V_n = U_n \quad V_n' = U_n'$$

à la surface de C. On aura alors:

$$V_{n+1} > V_n \quad V_{n+1}' < V_n' \quad V_n' > V_n.$$

Nous conclurons de là que V_n tend vers une limite finie et déterminée V.

Le théorème de Harnack montre que l'on a:

$$\Delta V = 0. \quad \text{D'ailleurs } V_n' > V > V_n.$$

Il reste à montrer que V tend vers U quand le point M se rapproche indéfiniment d'un point M_0 de la surface de C. Pour cela, il faut que dans le voisinage de M_0 la fonction U soit continue, sans quoi la proposition qu'il s'agit de démontrer serait fausse et n'aurait même pas de sens.

Nous voulons démontrer que l'on peut prendre M assez voisin de M_0 pour que

$$U - \varepsilon < V < U + \varepsilon$$

quelque petit que soit ε.

Comme au point M_0, U est continu, U_n et U'_n tendent vers une limite commune U quand n croît indéfiniment. Nous pourrons donc prendre n assez grand pour que :

$$U'_n - U_n < \frac{\varepsilon}{2}$$

d'où *a fortiori*

$$U'_n - U < \frac{\varepsilon}{2} \qquad U - U_n < \frac{\varepsilon}{2}.$$

Nous regardons n comme désormais déterminé, nous pourrons prendre M assez voisin de M_0 pour que :

$$V_n > U_n - \frac{\varepsilon}{2} \qquad V'_n < U'_n + \frac{\varepsilon}{2}.$$

On a alors :

$$V > V_n > U_n - \frac{\varepsilon}{2} > U - \varepsilon$$

et

$$V < V'_n < U'_n + \frac{\varepsilon}{2} < U + \varepsilon. \qquad \text{C. Q. F. D.}$$

§2.—*Problème de Fourier.*

Le problème de Fourier a pour objet l'étude du refroidissement d'un corps solide rayonnant. J'ai donné de ce problème, dans une note insérée aux Comptes Rendus, une solution plus rigoureuse et plus complète que celles qui ont été proposées jusqu'ici. Bien qu'elle ne soit pas encore entièrement satisfaisante, je crois qu'il ne sera pas inutile de la rappeler et de la développer ici ; car elle va nous servir de point de depart naturel pour ce qui va suivre.

Considérons un corps solide homogene et isotrope, isolé dans un milieu indéfini à travers lequel la chaleur se propage par rayonnement. Soit V la température d'un point du corps ; ce sera une fonction de x, y, z et t ; soit 0 la température extérieure. On aura, à l'intérieur du corps :

$$\frac{dV}{dt} = a^2 \Delta V \tag{1}$$

et à la surface du corps $$\frac{dV}{dn} + hV = 0. \tag{2}$$

a^2 est un coefficient constant qui dépend de la conductibilité du corps et de sa chaleur spécifique. Quant à h c'est un coefficient positif et constant qui dépend du pouvoir émissif du corps.

Le premier point est d'établir l'existence d'une infinité de fonctions auxiliaires $$U_1,\ U_2,\ \ldots,\ U_n,\ \ldots$$

ne dépendant que de x, y, z et satisfaisant aux équations suivantes:

On aura à l'intérieur du corps:

$$\Delta U_n + k_n U_n = 0$$

et à la surface: $$\frac{dU_n}{dn} + hU_n = 0.$$

Les quantités $$k_1,\ k_2,\ \ldots,\ k_n,\ \ldots$$

sont des coefficients constants que je supposerai rangés par ordre de grandeur croisante et que je déterminerai plus complètement dans la suite.

Enfin pour achever de définir la fonction U_n, nous ajouterons que l'intégrale

$$\int U_n^2 d\tau$$

étendue à tous les éléments de volume $d\tau$ du corps solide doit être égale à 1. Nous allons pour démontrer l'existence des fonctions U_n employer une démonstration analogue à celle par laquelle Riemann établit le principe de Dirichlet.

Soit F une fonction quelconque et posons:

$$A = \int F^2 d\tau,$$
$$B = h\int F^2 d\omega + \int\left[\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2\right] d\tau.$$

L'intégrale A ainsi que la seconde des intégrales de l'expression B sont étendues à tous les éléments $d\tau$ du volume du solide et l'intégrale $\int F^2 d\omega$ à tous les éléments $d\omega$ de sa surface.

Supposons que la fonction F soit assujettie à la condition

$$A = 1.$$

Les deux intégrales de l'expression B ont tous leurs éléments positifs. B ne peut donc devenir négatif. B ne peut non plus devenir nul; en effet il ne pourrait s'annuler que si tous les éléments des deux intégrales étaient nuls à la fois, c'est-à-dire si l'on avait:

$$F = 0$$

à la surface du corps et

$$\frac{dF}{dx} = \frac{dF}{dy} = \frac{dF}{dz} = 0$$

à l'intérieur du corps. Il faudrait donc que F fût encore nul à l'intérieur du corps, ce qui est impossible à cause de la condition:

$$A = \int F^2 d\tau = 1.$$

B admettra donc un minimum absolu. Soit U_1 la valeur de F qui correspond à ce minimum. Le calcul des variations nous donne:

$$\frac{1}{2}\,\delta A = \int U_1 \delta U_1 d\tau = 0,$$

$$\frac{1}{2}\,\delta B = h\int U_1 \delta U_1 d\omega + \int\left(\frac{dU_1}{dx}\frac{d\delta U_1}{dx} + \frac{dU_1}{dy}\frac{d\delta U_1}{dy} + \frac{dU_1}{dz}\frac{d\delta U_1}{dz}\right) d\tau = 0.$$

Or le théorème de Green nous donne:

$$\int\left(\frac{dU_1}{dx}\frac{d\delta U_1}{dx} + \frac{dU_1}{dy}\frac{d\delta U_1}{dy} + \frac{dU_1}{dz}\frac{d\delta U_1}{dz}\right) d\tau = \int \frac{dU_1}{dn}\,\delta U_1 d\omega - \int \Delta U_1 \delta U_1 d\tau$$

de sorte qu'il vient:

$$\frac{1}{2}\,\delta B = \int\left(hU_1 + \frac{dU_1}{dn}\right)\delta U_1 d\omega - \int \Delta U_1 \delta U_1 d\tau = 0.$$

δB doit s'annuler toutes les fois que δA s'annule. On doit donc d'après l'une des régles du calcul des variations, pouvoir trouver une constante k_1 telle que

$$\delta B - k_1 \delta A$$

soit nulle quel que soit δU_1. On doit donc avoir identiquement:

$$\int\left(\frac{dU_1}{dn} + hU_1\right)\delta U_1 d\omega - \int(\Delta U_1 + k_1 U_1)\,\delta U_1 d\tau = 0$$

ce qui exige que tous les éléments des deux intégrales soient nuls ou que l'on ait à la surface du corps

$$\frac{dU_1}{dn} + hU_1 = 0$$

et à l'intérieur: $$\Delta U_1 + k_1 U_1 = 0.$$
On a d'ailleurs par hypothèse:
$$A = \int U_1^2 d\tau = 1.$$

L'existence de la fonction U_1 est donc démontrée.

On trouve:
$$B = h\int U_1^2 d\omega + \int\left[\left(\frac{dU_1}{dx}\right)^2 + \left(\frac{dU_1}{dy}\right)^2 + \left(\frac{dU_1}{dz}\right)^2\right] d\tau$$
ou en vertu du théorème de Green:
$$B = \int U_1\left(\frac{dU_1}{dn} + hU_1\right) d\omega - \int U_1 \Delta U_1 d\tau$$
ou en vertu des équations qui défininent U_1:
$$B = k_1 \int U_1^2 d\tau = k_1.$$
Ainsi k_1 n'est autre chose que la valeur du rapport $\frac{B}{A}$ pour $F = U_1$. Comme ce rapport atteint son minimum pour $F = U_1$, nous devons conclure que k_1 est le minimum du rapport $\frac{B}{A}$.

Prenons pour F une valeur quelconque, nous obtiendrons une certaine valeur de $\frac{B}{A}$ qui sera plus grande que k_1. C'est donc un moyen de trouver une limite supérieure de k_1.

Faisons par exemple $$F = 1.$$
Il vient:
$$A = \int d\tau = \text{volume du corps solide},$$
$$B = h\int d\omega = h \times \text{surface du corps solide}.$$
Le rapport $\frac{k_1}{h}$ est donc toujours plus petit que le rapport de la surface du solide à son volume.

Cherchons encore une autre inégalité.

Appelons W le volume du corps solide et S sa surface.

Prenons pour origine des coordonnées le centre de gravité du volume du corps.

Appelons I l'intégrale
$$\int x^2 d\tau\,.$$
c'est-à-dire le moment d'inertie du volume par rapport au plan des yz.

Appelons H l'intégrale
$$\int x^2 d\omega$$
c'est-à-dire le moment d'inertie de la surface par rapport au plan des yz. Soit x_0 la distance du centre de gravité de la surface au plan des yz. Posons:
$$M = x_0 S = \int x d\omega.$$
Faisons maintenant:
$$F = \alpha x + 1,$$
α étant une indéterminée; il viendra:
$$A = \int (\alpha x + 1)^2 d\tau = \alpha^2 I + W,$$
$$B = h \int (\alpha x + 1)^2 d\omega + \int \alpha^2 d\tau = \alpha^2 (hH + W) + 2\alpha M h + Sh$$
d'où:
$$k_1 < \frac{\alpha^2 (hH + W) + 2\alpha M h + Sh}{\alpha^2 I + V}.$$

Il faut maintenant choisir α de telle sorte que le second membre de cette inégalité soit minimum. Ce second membre admet un maximum et un minimum qui sont les racines de l'équation en λ:
$$M^2 h^2 = (hH + W - \lambda I)(Sh - \lambda W)$$
ou
$$\lambda^2 IV - \lambda (ISh + W^2 + WHh) + SHh^2 - M^2 h^2 + WSh = 0.$$

On a donc:
$$k_1 < \frac{ISh + W^2 + WHh}{2IW} - \frac{\sqrt{(ISh + WHh + W^2)^2 - SHh^2 + M^2 h^2 - WSh}}{2IW}$$
cette limite est inférieure à la précédente.

Occupons nous maintenant de démontrer l'existence de la fonction U_2; soit une fonction quelconque F assujettie aux deux conditions suivantes:
$$A = 1\,,\quad C = \int F U_1 d\tau = 0.$$

On pourra choisir cette fonction de façon que B soit minimum; soit U_2 la fonction qu'il faut choisir pour F afin de rendre B minimum. On devra avoir:
$$\frac{1}{2}\,\delta B = \int \left(h U_2 + \frac{dU_2}{dn} \right) \delta U_2 d\omega - \int \Delta U_2 \delta U_2 d\tau = 0$$

toutes les fois que:

$$\frac{1}{2}\,\delta A=\int U_2\delta U_2 d\tau=0 \qquad \delta C=\int U_1\delta U_2 d\tau=0.$$

On pourra donc trouver deux constantes λ_1 et k_2 telles que:

$$\delta B-k_2\delta A-2\lambda_1\delta C=0$$

quelque soit δU_2. On a donc identiquement:

$$\int\left(hU_2+\frac{dU_2}{dn}\right)\delta U_2 d\omega-\int(\Delta U_2+k_2U_2+\lambda_1U_1)\,\delta U_2 d\tau=0$$

de sorte qu'on aura à la surface du corps

$$hU_2+\frac{dU_2}{dn}=0$$

et à l'intérieur:

$$\Delta U_2+k_2U_2+\lambda_1U_1=0.$$

Calculons λ_1 et k_2. Nous trouvons d'abord:

$$\int U_1\Delta U_2 d\tau+k_2\int U_1U_2 d\tau+\lambda_1\int U_1^2 d\tau=0.$$

Or

$$\int U_1^2 d\tau=1 \qquad C=\int U_1U_2 d\tau=0,$$

on aura donc:

$$\lambda_1+\int U_1\Delta U_2 d\tau=0.$$

Or le théorème de Green donne:

$$\int U_1\Delta U_2 d\tau-\int U_2\Delta U_1 d\tau=\int U_1\frac{dU_2}{dn}\,d\omega=\int U_2\frac{dU_1}{dn}\,d\omega.$$

Mais

$$\frac{dU_2}{dn}=-hU_2, \qquad \frac{dU_1}{dn}=-hU_1.$$

Il reste donc simplement:

$$\int U_1\Delta U_2 d\tau=\int U_2\Delta U_1 d\tau.$$

Mais $\Delta U_1=-k_1U_1.$

Donc

$$\lambda_1=\int U_1\Delta U_2 d\tau=-k_1\int U_1U_2 d\tau=0.$$

Donc λ_1 est nul.

Il vient donc:

$$\Delta U_2+k_2U_2=0$$

d'où:

$$\int U_2\Delta U_2 d\tau+k_2\int U_2^2 d\tau=0.$$

Or

$$A = \int U_2^2 d\tau = 1.$$

Donc:

$$k_2 = -\int U_2 \Delta U_2 d\tau = -\int U_2 \frac{dU_2}{dn} d\omega + \int \left[\left(\frac{dU_2}{dx}\right)^2 + \left(\frac{dU_2}{dy}\right)^2 + \left(\frac{dU_2}{dz}\right)^2 \right] d\tau$$

ou:

$$k_2 = h \int U_2^2 d\omega + \int \Sigma \left(\frac{dU_2}{dx}\right)^2 d\tau = B.$$

Ainsi k_2 n'est autre chose que la valeur de B qui correspond à $F = U_2$. k_1 était la valeur de B qui correspond à $F = U_1$ et au minimum de B.

Par conséquent on a:

$$k_1 < k_2$$

et d'ailleurs:

$$\frac{dU_2}{dn} + hU_2 = 0 \quad \text{à la surface du corps,}$$

$$\Delta U_2 + k_2 U_2 = 0 \quad \text{à l'intérieur,}$$

$$\int U_2^2 d\tau = 1 \quad \int U_1 U_2 d\tau = 0.$$

L'existence de la fonction U_2 est donc démontrée.

Soit maintenant U_3 une fonction telle que

$$A = \int U_3^2 d\tau = 1, \quad C = \int U_3 U_1 d\tau = 0, \quad D = \int U_3 U_2 d\tau = 0$$

et choisie d'ailleurs de telle sorte que B soit aussi petit que possible.

On devra avoir: $$\delta B = 0$$
toutes les fois que

$$\delta A = 0, \quad \delta C = 0, \quad \delta D = 0.$$

On pourra donc trouver trois constantes λ_1, λ_2 et k_3 telles que l'on ait identiquement:

$$\delta B - k_3 \delta A - 2\lambda_1 \delta C - 2\lambda_2 \delta D = 0.$$

Un raisonnement analogue à celui qui précède montrerait que l'on doit avoir à la surface du corps

$$\frac{dU_3}{dn} + hU_3 = 0$$

et à l'intérieur

$$\Delta U_3 + k_3 U_3 + \lambda_1 U_1 + \lambda_2 U_2 = 0.$$

On demontrerait ensuite comme on l'a fait plus haut que λ_1 et λ_2 sont nuls et que k_3 est la valeur de B qui correspond à $F = U_3$.

D'après la définition de U_3, k_3 est donc la plus petite valeur que puisse prendre l'expression B quand la fonction F est assujettie aux conditions:

$$\int F^2 d\tau = 1; \quad \int F U_1 d\tau = 0; \quad \int F U_2 d\tau = 0.$$

D'autre part k_2 était la plus petite valeur que pouvait prendre B quand F était assujettie aux deux premières de ces conditions seulement. Donc:

$$k_3 > k_2.$$

La fonction U_3 est ainsi définie par les conditions:

$$\frac{dU_3}{dn} + hU_3 = 0, \quad \Delta U_3 + k_3 U_3 = 0,$$
$$\int U_3^2 d\tau = 1; \int U_3 U_1 d\tau = \int U_3 U_2 d\tau = 0.$$

La loi est manifeste; il est inutile de pousser plus loin ce raisonnement. On voit que l'on a démontré l'existence d'une infinité de fonctions:

$$U_1, \ U_2, \ \ldots\ldots, \ U_p, \ \ldots.$$

telles que l'on a à la surface du corps

$$\frac{dU_p}{dn} + hU_p = 0$$

et à l'intérieur:
$$\Delta U_p + k_p U_p = 0.$$

Les coefficients k_p sont des constantes positives et telles que:

$$k_{p+1} > k_p.$$

Enfin on a:
$$\int U_p U_q d\tau = 0 \qquad \text{pour } p \gtrless q$$

et
$$\int U_p^2 d\tau = 1.$$

Ce raisonnement est sujet aux mêmes objections que celui par lequel Riemann établit le principe de Dirichlet. Nous nous en contenterons toutefois pour le moment, nous chercherons dans la suite à le rendre plus rigoureux.

Ces fonctions U_p ont été entièrement construites par Lamé dans certains cas particuliers, par exemple dans celui de la sphère et celui du parallélipipède.

Dans celui du parallélipipède rectangle dont les trois dimensions parallèles aux trois axes sont $2a$, $2b$ et $2c$, l'expression des fonctions U_p et des coefficients k_p est particulièrement simple.

Nous devons avoir en effet à l'intérieur du corps:

$$\Delta U + kU = 0$$

et en outre:

$$\text{pour } x = a \quad \frac{dU}{dx} + hU = 0, \quad \text{pour } x = -a \quad \frac{dU}{dx} - hU = 0,$$
$$\text{pour } y = b \quad \frac{dU}{dy} + hU = 0, \quad \text{pour } y = -b \quad \frac{dU}{dy} - hU = 0;$$
$$\text{pour } z = c \quad \frac{dU}{dz} + hU = 0, \quad \text{pour } z = -c \quad \frac{dU}{dz} - hU = 0.$$

Posons:

$$U = \sin(\lambda_1 x + \mu_1) \sin(\lambda_2 x + \mu_2) \sin(\lambda_3 x + \mu_3).$$

Les constantes λ_1 et μ_1 nous seront données par les deux équations:

$$\lambda_1 \cos(\lambda_1 a + \mu_1) + h \sin(\lambda_1 a + \mu_1) = 0,$$
$$\lambda_1 \cos(\lambda_1 a - \mu_1) + h \sin(\lambda_1 a - \mu_1) = 0,$$

d'où:

$$\operatorname{tg}(\lambda_1 a + \mu_1) = \operatorname{tg}(\lambda_1 a - \mu_1),$$
$$\mu_1 = \frac{\varkappa\pi}{2},$$

$\varkappa$ étant un entier; il suffira de prendre $\varkappa = 0$ d'où:

$$\sin(\lambda_1 x + \mu_1) = \sin \lambda_1 x$$

et $\varkappa = 1$ d'où:

$$\sin(\lambda_1 x + \mu_1) = \cos \lambda_1 x.$$

On aura alors pour

$$\mu_1 = 0 \text{ ou } \frac{\pi}{2} \qquad \lambda_1 + h \operatorname{tg}(\lambda_1 a + \mu_1) = 0.$$

De même μ_2, μ_3, λ_2 et λ_3 seront données par les équations:

$$\mu_2 = 0 \text{ ou } \frac{\pi}{2} \qquad \lambda_2 + h \operatorname{tg}(\lambda_2 b + \mu_2) = 0.$$

$$\mu_3 = 0 \text{ ou } \frac{\pi}{2} \qquad \lambda_3 + h \operatorname{tg}(\lambda_3 c + \mu_3) = 0.$$

Enfin on a:

$$k = \lambda_1^2 + \lambda_2^2 + \lambda_3^2.$$

Considérons en particulier le cas de $h = 0$; ce qui correspond au cas où la surface du corps est imperméable à la chaleur; il vient

$$\operatorname{tg}(\lambda_1 a + \mu_1) = \infty$$

d'où:

$$\lambda_1 a + \mu_1 = \frac{\pi}{2} + \varkappa\pi \qquad \varkappa \text{ étant entier:}$$

ou à cause de $\mu_1 = 0$ ou $\frac{\pi}{2}$:

$$\lambda_1 = \frac{m_1 \pi}{2a} \quad m_1 \text{ étant entier.}$$

On a alors:

$$k = \left(\frac{m_1^2}{a^2} + \frac{m_2^2}{b^2} + \frac{m_3^2}{c^2}\right) \frac{\pi^2}{4}$$

où m_1, m_2, m_3 sont des entiers. On aura donc:

$$k_1 = 0, \quad k_2 = \frac{\pi^2}{4a^2}$$

si $2a$ est la plus grande des trois dimensions du parallélipipède, c'est-à-dire si

$$a > b > c.$$

On trouve ainsi $\quad k_1 = 0, \quad U_1 = \text{const.}$

et $\quad k_2 = \frac{\pi^2}{4a^2}, \quad U_2 = \text{const. } \sin \lambda_1 x.$

Supposons maintenant $h = \infty$, ce qui correspond au cas où la surface du corps est maintenue artificiellement à la température 0; on a alors:

$$\operatorname{tg}(\lambda_1 a + \mu_1) = 0$$

d'où $\quad \lambda_1 a + \mu_1 = \varkappa\pi, \quad \varkappa$ étant entier,

ou puisque $\mu_1 = 0$ ou $\frac{\pi}{2}$,

$$\lambda_1 a = m_1 \frac{\pi}{2}, \quad m_1 \text{ étant entier.}$$

Nous trouvons donc encore:

$$k = \left(\frac{m_1^2}{a^2} + \frac{m_2^2}{b^2} + \frac{m_3^2}{c^2}\right) \frac{\pi^2}{4},$$

m_1, m_2 et m_3 étant entiers.

Mais toutes les solutions ne sont pas acceptables. On doit avoir en effet

$$U = 0$$

à la surface du corps; d'où:

$$\sin(\lambda_1 a + \mu_1) = \sin(\lambda_2 b + \mu_2) = \sin(\lambda_3 c + \mu_3) = 0,$$

ce qui exige que m_1, m_2 et m_3 soient au moins égaux à 1; il viendra donc:

$$k_1 = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \frac{\pi^2}{4}.$$

Laissons maintenant de côté le cas du parallélipipède rectangle et revenons au

cas général; il est clair que, pour un même corps, k_1, k_2, ... k_n sont des fonctions du pouvoir émissif h. Je dis que ces fonctions sont toujours croissantes. Soient en effet h' et h'' deux valeurs du pouvoir émissif; soient pour un corps donné, k'_n et k''_n les valeurs correspondantes de k_n, et U'_n et U''_n les fonctions U_n correspondantes. On aura:

à la surface du corps

$$\frac{dU'_n}{dn} + h'U'_n = \frac{dU''_n}{dn} + h''U''_n = 0 \tag{3}$$

et à l'intérieur du corps:

$$\Delta U'_n + k'_n U'_n = \Delta U''_n + k''_n U''_n = 0. \tag{4}$$

Le théorème de Green nous donne:

$$\int \left(\frac{dU'_n}{dn} U''_n - \frac{dU''_n}{dn} U'_n \right) d\omega = \int (U''_n \Delta U'_n - U'_n \Delta U''_n)\, d\tau.$$

Dans cette identité remplaçons $\Delta U'_n$, $\Delta U''_n$, $\frac{dU'_n}{dn}$, $\frac{dU''_n}{dn}$ par leurs valeurs tirées des équations (2) et (4), il viendra:

$$(h' - h'') \int U'_n U''_n d\omega = (k'_n - k''_n) \int U'_n U''_n d\tau.$$

Supposons que h' et h'' diffère très peu de telle sorte que

$$h'' - h' = dh, \quad k''_n - k'_n = dk_n,$$

$U'_n = U''_n$ à des infiniments petits près, il vient:

$$dh \int U_n'^2 d\omega = dk_n \int U_n'^2 d\tau. \tag{5}$$

Dans l'équation (5), les deux intégrales sont essentiellement positives; il reste donc:

$$\frac{dk_n}{dh} > 0,$$

ce qui signifie que k_n est une fonction croissante de h. C. Q. F. D.

Pour $h = 0$; on a évidemment:

$$k_1 = 0, \quad U_1 = \text{const.} = \frac{1}{\sqrt{W}} \qquad (W \text{ étant le volume du corps}).$$

En effet, ces quantités satisfont aux équations

$$\Delta U_1 = \Delta U_1 + k_1 U_1 = 0, \quad \frac{dU_1}{dn} = 0.$$

L'équation (5) devient alors

$$\frac{dk_1}{dh} = \frac{S}{W},$$

S désignant la surface du corps et W son volume. Donc si h est très petit, k_1 est sensiblement égal à

$$h\frac{S}{W}.$$

Nous venons de trouver:

$$\frac{dh}{dk_n}\int U_n^2 d\omega = \int U_n^2 d\tau = 1.$$

Nous avons d'autre part:

$$k_n = h\int U_n^2 d\omega + \int\left[\left(\frac{dU_n}{dx}\right)^2 + \left(\frac{dU_n}{dy}\right)^2 + \left(\frac{dU_n}{dz}\right)^2\right] d\tau$$

d'où:

$$k_n > h\int U_n^2 d\omega, \quad \int U_n^2 d\omega < \frac{k_n}{h},$$

$$1 < \frac{k_n}{h}\,\frac{dh}{dk_n}$$

et

$$\frac{dk_n}{k_n} < \frac{dh}{h}.$$

Cette inegalité montre que le rapport $\frac{k_n}{h}$ va toujours en décroissant.

Il importe de remarquer, avant d'aller plus loin que quand h est positif, k_n est essentiellement positif. Sans doute, cela résulte de la façon dont les fonctions U_n ont été définies plus haut; mais on pourrait imaginer qu'il existe des fonctions U autres que celles que nous venons de définir et pour lesquelles on aurait:

$$\frac{dU}{dn} + hU = 0, \quad \Delta U + kU = 0, \quad \int U^2 d\tau = 1, \quad k < 0.$$

Je dis que cela est impossible et il me suffit pour l'établir de montrer que l'on a:

$$k = h\int U^2 d\omega + \int\left[\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 + \left(\frac{dU}{dz}\right)^2\right] d\tau,$$

ce qui se démontrerait par le même calcul que plus haut.

Si au contraire h était négatif, k_n pourrait aussi devenir négatif.

Il peut arriver quand on fait varier h, que deux des nombres k_n et k_{n+1} viennent à se confondre. Qu'arrivera-t-il alors en général?

Soient k' et k'' deux valeurs de k, U' et U'' les fonctions U correspondantes. Imaginons que k', k'', U' et U'' soient des fonctions continues de h. Quand $h < h_0$, on aura par exemple $k' < k''$; pour $h = h_0$, on aura $k' = k''$; pour $h > h_0$,

on aura $k' > k''$. D'après ce que nous venons de démontrer, k' et k'' sont deux fonctions croissantes de h.

Maintenant, supposons qu'il y ait $n-1$ nombres k inférieurs à la fois à k' et à k''. Nous devrons, d'après nos conventions, appeler k_n la plus petite et k_{n+1} la plus grande des deux quantités k' et k''. Nous aurons donc:

$$k' = k_n \quad k'' = k_{n+1} \quad \text{pour } h < h_0$$

et

$$k'' = k_n \quad k' = k_{n+1} \quad \text{pour } h > h_0.$$

Les fonctions k' et k'' étant continues et croissantes, les fonctions k_n et k_{n+1} définies comme nous venons de le faire seront aussi continues et croissantes.

En résumé, dans tous les cas possibles, k_n est une fonction croissante de h; k_n atteint donc sa plus petite valeur pour $h = 0$.

Nous allons donc étudier la valeur de k_n pour $h = 0$.

Décomposons le volume de notre corps solide, d'une manière quelconque, en p volumes partiels. Considérons chacun de ces volumes partiels comme un corps solide de même conductibilité que le solide donné et dont la surface est imperméable à la chaleur. Soient:

$$\begin{array}{l} U_{11}\ ,\ U_{12}\ ,\ \ldots\ldots,\ U_{1n}\ ,\ \ldots\ldots, \\ U_{21}\ ,\ U_{22}\ ,\ \ldots\ldots,\ U_{2n}\ ,\ \ldots\ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ U_{p1}\ ,\ U_{p2}\ ,\ \ldots\ldots,\ U_{pn}\ ,\ \ldots\ldots, \end{array}$$

les fonctions U relatives à ces p solides.

Soient:

$$\begin{array}{l} k_{11}\ ,\ k_{12}\ ,\ \ldots\ldots,\ k_{1n}\ ,\ \ldots\ldots, \\ k_{21}\ ,\ k_{22}\ ,\ \ldots\ldots,\ k_{2n}\ ,\ \ldots\ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ k_{p1}\ ,\ k_{p2}\ ,\ \ldots\ldots,\ k_{pn}\ ,\ \ldots\ldots, \end{array}$$

les nombres k correspondants.

Comme pour chacun des p solides partiels on a $h = 0$, on aura:

$$k_{11} = k_{21} = \ldots\ldots = k_{p1} = 0$$

et les p fonctions $U_{11}, U_{21}, \ldots\ldots, U_{p1}$ seront des constantes.

Je conserverai la notation U_n et k_n pour les fonctions relatives au solide total. Posons maintenant:

$$V = \alpha_1 U_1 + \alpha_2 U_2 + \ldots\ldots + \alpha_n U_n,$$

les α étant des coefficients indéterminés.

Nous avons à la surface du corps, puisque h est supposé nul:

$$\frac{dV}{dn}=0$$

et par conséquent en vertu du théorème de Green:

$$\int\left[\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right]d\tau=-\int V\Delta V d\tau.$$

Le second membre peut s'écrire (en vertu de l'équation $\Delta U_i+k_iU_i=0$):

$$\int(\alpha_1U_1+\alpha_2U_2+\ldots\ldots+\alpha_nU_n)(\alpha_1k_1U_1+\alpha_2k_2U_2+\ldots\ldots+\alpha_nk_nU_n)\,d\tau$$

ou encore $\left(\text{en vertu des relations} \int U_i^2d\tau=1,\quad \int U_iU_kd\tau=0\right)$:

$$k_1\alpha_1^2+k_2\alpha_2^2+\ldots\ldots+k_n\alpha_n^2.$$

D'autre part on a:

$$\int V^2d\tau=\int(\alpha_1U_1+\alpha_2U_2+\ldots\ldots+\alpha_nU_n)^2d\tau=\alpha_1^2+\alpha_2^2+\ldots\ldots+\alpha_n^2,$$

d'où

$$\frac{\int\left[\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right]d\tau}{\int V^2d\tau}=\frac{k_1\alpha_1^2+k_2\alpha_2^2+\ldots\ldots+k_n\alpha_n^2}{\alpha_1^2+\alpha_2^2+\ldots\ldots+\alpha_n^2}<k_n. \tag{6}$$

Nous pouvons disposer de nos n coefficients arbitraires $\alpha_1,\alpha_2,\ldots\ldots,\alpha_n$ de façon à satisfaire à $n-1$ conditions. Voici comment nous choisirons ces conditions.

Nous annulerons d'abord les λ_1 intégrales

$$\int VU_{11}d\tau,\quad \int VU_{12}d\tau,\ \ldots\ldots,\quad \int VU_{1\lambda_1}d\tau$$

étendues au premier solide partiel.

Nous annulerons ensuite les λ_2 intégrales

$$\int VU_{21}d\tau,\ \ldots\ldots,\quad \int VU_{2\lambda_2}d\tau$$

étendues au second solide partiel.

Nous annulerons enfin les λ_p intégrales

$$\int VU_{p1}d\tau,\ \ldots\ldots,\quad \int VU_{p\lambda_p}d\tau$$

étendues au dernier solide partiel.

Nous devrons avoir d'ailleurs :

$$\lambda_1 + \lambda_2 + \ldots + \lambda_p = n - 1. \tag{7}$$

Rappelons quelle est la définition de $k_{1.\lambda_1+1}$. Ce coefficient est le minimum du rapport des deux intégrales :

$$B = \int \left[\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right] d\tau, \quad A = \int V^2 d\tau;$$

quand la fonction arbitraire V est assujettie aux λ_1 conditions :

$$\int VU_{11} d\tau = \int VU_{12} d\tau = \ldots = \int VU_{1\lambda_1} d\tau = 0.$$

Toutes les intégrales sont supposées étendues au premier solide partiel. Donc le rapport des deux intégrales B et A étendues à ce premier solide est plus grand que $k_{1.\lambda_1+1}$.

On verrait de même que le rapport de ces deux intégrales étendues au second solide est plus grand que $k_{2.\lambda_2+1}$, etc.; que le rapport de ces deux intégrales étendues au p^e solide partiel est plus grand que $k_{p.\lambda_p+1}$.

Donc le rapport des deux intégrales B et A étendues au solide total, sera plus grand que la plus petite des p quantités :

$$k_{1.\lambda_1+1}, \; k_{2.\lambda_2+1}, \; \ldots, \; k_{p.\lambda_p+1}. \tag{8}$$

Mais l'inégalité (6) exprime que ce même rapport est plus petit que k_n.

Donc k_n est plus grand que la plus petite des quantités (8) pourvu que la relation (7) soit satisfaite.

Ce résultat peut s'énoncer comme il suit :

Rangeons les quantités

$$\begin{array}{l} k_{11}, \; k_{12}, \; \ldots, \; k_{1n}, \; \ldots \\ k_{21}, \; \ldots \ldots, \; k_{2n}, \; \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ k_{p1}, \; \ldots\ldots\ldots, \; k_{pn}, \; \ldots \end{array}$$

par ordre de grandeur croissante. (Il va sans dire que dans la série ainsi obtenue, plusieurs termes pourront être égaux ; c'est ainsi que les p premiers termes qui sont $k_{11}, k_{21}, \ldots, k_{p1}$ sont tous égaux à 0.)

Chacun des termes de la série ainsi obtenue sera plus petit que le terme correspondant de la série :

$$k_1, \; k_2, \; \ldots, \; k_n, \; \ldots.$$

Cela posé, supposons que notre solide soit un polyèdre limité par des faces qui soient toutes parallèles à l'un des trois plans de coordonnées.

Nous pourrons décomposer notre polyèdre en $n-1$ parallélipipèdes rectangles. Soient

$$k_{12},\ k_{22},\ \ldots\ldots,\ k_{n-1.2} \tag{9}$$

les valeurs de k_2 correspondant à ces $n-1$ parallélipipèdes; d'après ce que nous venons de voir, la valeur de k_n correspondant au polyèdre total, sera plus grande que la plus petite des quantités (9).

Si les trois dimensions d'un parallélipipède sont $2a$, $2b$, $2c$ de telle sorte que $a > b > c$; nous avons vu qu'on a, pour ce parallélipipède:

$$k_2 = \frac{\pi^2}{4a^2}.$$

Si donc aucune des trois dimensions d'aucun de nos $n-1$ parallélipipèdes partiels n'excède a, on aura pour le polyèdre total

$$k_n > \frac{\pi^2}{4a^2}.$$

Quand n croît indéfiniment, on peut faire tendre a vers 0; donc k_n croît indéfiniment.

Cela est vrai pour un polyèdre formé comme je viens de le dire; et comme on peut construire de pareils polyèdres qui diffèrent aussi peu qu'on le veut d'un solide quelconque, on pourrait dire que cela doit être vrai aussi d'un solide quelconque. Mais un semblable raisonnement ne saurait nous contenter.

Pour démontrer le théorème pour un solide quelconque, je dois d'abord chercher une limite inférieure de k_2 pour un solide *convexe* quelconque. Par solide convexe, j'entends un corps tel que le segment de droite MM', qui joint deux points M et M' intérieurs au corps, soit toujours tout entier intérieur au corps; ou ce qui revient au même tel qu'aucune droite ne rencontre la surface du corps en plus de deux points.

Rappelons d'abord la définition de k_2; k_2 est le minimum du rapport

$$\frac{\int\left[\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right]d\tau}{\int V^2 d\tau}$$

quand la fonction V et assujettie à la condition:

$$\int V d\tau = 0. \tag{10}$$

On peut transformer la définition de façon à faire disparaître cette dernière condition.

Envisageons l'intégrale

$$\int (V - V')^2 d\tau\, d\tau'.$$

Dans cette intégrale, $d\tau$ et $d\tau'$ sont deux éléments de volume quelconques du solide donné; x, y, z, et x', y', z' sont les coordonnées du centre de gravité de chacun de ces deux éléments; V et V' sont les valeurs de la fonction V aux points x, y, z et x', y', z'; enfin l'intégrale est étendue à toutes les combinaisons de deux éléments de volume $d\tau$ et $d\tau'$. (Chaque combinaison sera répétée deux fois de la façon suivante; une première fois le premier élément jouera le rôle de $d\tau$ et le second celui de $d\tau'$ et la seconde fois ce sera le contraire.)

On trouve aisément en se servant de la relation (10) et en appelant W le volume total du corps

$$\int (V - V')^2 d\tau d\tau' = 2W \int V^2 d\tau$$

de sorte que k_2 sera le minimum de l'expression

$$\frac{2W\int\left[\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right]d\tau}{\int (V - V')^2 d\tau\, d\tau'}. \tag{11}$$

Mais alors la condition (10) devient inutile; si en effet on ajoute à V une constante quelconque, la condition (10) cesse d'être satisfaite et l'expression (11) ne change pas.

Nous pouvons donc dire finalement que k_2 est le minimum de l'expression (11), la fonction V étant tout à fait quelconque et n'étant assujettie à aucune condition.

C'est de la transformation de cette expression (11) que nous allons maintenant nous occuper.

Posons à cet effet:

$$\begin{aligned} x &= \xi + \rho \cos\phi \sin\theta, & x' &= \xi + \rho' \cos\phi \sin\theta, \\ y &= \eta + \rho \sin\phi \sin\theta, & y' &= \eta + \rho' \sin\phi \sin\theta, \\ z &= \rho \cos\theta \quad , & z' &= \rho' \cos\theta. \end{aligned} \tag{12}$$

La théorie de la transformation des intégrales multiples nous donne:

$$\int (V - V')^2 d\tau d\tau' = \int (V - V')^2 dx dy dz dx' dy' dz'$$

$$= \int (V - V')^2 (\rho - \rho')^2 \sin\theta \cos\theta d\xi d\eta d\rho d\rho' d\theta d\phi.$$

Il va sans dire que, bien que je n'emploie qu'un seul signe $\int$, il s'agit ici d'intégrales sextuples.

Parlons maintenant des limites d'intégration; supposons que ξ, η, θ et ϕ soient considérés un instant comme des constantes; quand on fera varier ρ, le point x, y, z décrira une droite; comme le corps est convexe, cette droite rencontrera la surface du corps en deux points. Soient ρ_0 et ρ_1 les valeurs de ρ qui correspondent à ces deux points d'intersection.

Pour obtenir alors tous les points (x, y, z), (x', y', z') intérieurs au corps il faudra faire varier ρ et ρ' de ρ_0 à ρ_1, θ de 0 à $\frac{\pi}{2}$, ϕ de 0 à 2π, et donner à ξ et à η toutes les valeurs telles que ρ_0 et ρ_1 soient réels.

Cela posé, cherchons à transformer le numérateur de l'expression (11).

Nous avons d'abord, en vertu des équations (12):

$$\frac{dV}{d\rho} = \cos\phi \sin\theta \frac{dV}{dx} + \sin\phi \sin\theta \frac{dV}{dy} + \cos\theta \frac{dV}{dz}$$

d'où:

$$\int\int\left(\frac{dV}{d\rho}\right)^2 \sin\theta d\theta d\phi$$
$$= \int\int\left(\cos\phi \sin\theta \frac{dV}{dx} + \sin\phi \sin\theta \frac{dV}{dy} + \cos\theta \frac{dV}{dz}\right)^2 \sin\theta d\theta d\phi.$$

Calculons cette intégrale double en intégrant entre les limites 0 et 2π pour ϕ et entre les limites 0 et $\frac{\pi}{2}$ pour θ. Le coefficient de $\left(\frac{dV}{dx}\right)^2$ sera:

$$\int\int \cos^2\phi \sin^3\theta d\theta d\phi = \frac{2\pi}{3}.$$

Il est aisé de voir que le coefficient de $\left(\frac{dV}{dy}\right)^2$ a la même valeur.

Le coefficient de $\left(\frac{dV}{dz}\right)^2$ sera:

$$\int\int \cos^2\theta \sin\theta \, d\theta \, d\phi = \frac{2\pi}{3}.$$

Le coefficient de $2\frac{dV}{dx}\frac{dV}{dy}$ sera:

$$\int\int \cos\phi \sin\phi \sin^3\theta \, d\theta \, d\phi = 0.$$

Ceux de

$$2\,\frac{dV}{dx}\,\frac{dV}{dz} \text{ et de } 2\,\frac{dV}{dy}\,\frac{dV}{dz} \text{ seront :}$$

$$\int\!\!\int \cos\phi\,\sin^2\theta\,\cos\theta\,d\theta\,d\phi = \int\!\!\int \sin\phi\,\sin^2\theta\,\cos\theta\,d\theta\,d\phi = 0\,.$$

Il reste donc simplement :

$$\int\!\!\int \left(\frac{dV}{d\rho}\right)^2 \sin\theta\,d\theta\,d\phi = \frac{2\pi}{3}\left[\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right].$$

Soit M une fonction quelconque de x, y et z; la théorie de la transformation des intégrales multiples donne :

$$\int M d\tau = \int M dx\,dy\,dz = \int M \cos\theta\,d\xi\,d\eta\,d\rho\,.$$

Il vient donc :

$$\int \left(\frac{dV}{d\rho}\right)^2 \sin\theta\cos\theta\,d\xi\,d\eta\,d\rho\,d\theta\,d\phi = \frac{2\pi}{3}\int \left[\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right] d\tau.$$

Posons maintenant pour abréger :

$$B = \int_{\rho_0}^{\rho_1} \left(\frac{dV}{d\rho}\right)^2 d\rho, \quad A = \int_{\rho_0}^{\rho_1} d\rho \int_{\rho_0}^{\rho_1} d\rho'\,(V - V')^2(\rho - \rho')^2,$$

l'expression (11) deviendra :

$$\frac{3W}{\pi}\,\frac{\int B \sin\theta\,\cos\theta\,d\xi\,d\eta\,d\theta\,d\phi}{\int A \sin\theta\,\cos\theta\,d\xi\,d\eta\,d\theta\,d\phi}\,.$$

Les quantités sous les deux signes $\int$ sont essentiellement positives puisque θ varie entre 0 et $\frac{\pi}{2}$. Pour trouver une limite inférieure de l'expression (11), il nous suffira de connaître une limite inférieure du rapport $\frac{B}{A}$. C'est de quoi nous allons maintenant nous occuper.

Si la fonction V est choisie de telle façon que $A = 1$, il est clair que l'intégrale B ne pourra pas s'annuler ; elle admettra donc un certain minimum. Cherchons à déterminer ce minimum par le calcul des variations.

Nous trouvons :

$$\frac{1}{2}\,\delta B = \int_{\rho_0}^{\rho_1} \frac{dV}{d\rho}\,\frac{d\delta V}{d\rho}\,d\rho = 0,$$

$$\frac{1}{2}\,\delta A = \int\int (V - V')(\delta V - \delta V')(\rho - \rho')^2\,d\rho\,d\rho' = 0.$$

Transformons ces deux expressions, nous trouvons, par l'intégration par parties,

$$\frac{1}{2}\,\delta B = \left[\frac{dV}{d\rho}\,\delta V\right]_{\rho_0}^{\rho_1} - \int_{\rho_0}^{\rho_1} \frac{d^2 V}{d\rho^2}\,\delta V\,d\rho = 0.$$

D'autre part, si dans l'intégrale double

$$J = \int\int (V - V')\,\delta V\,(\rho - \rho')^2\,d\rho\,d\rho'$$

on permute ρ et ρ', l'intégrale ne doit pas changer il vient ainsi :

$$J = \int\int (V' - V)\,\delta V'\,(\rho' - \rho)^2\,d\rho\,d\rho',$$

d'où

$$\frac{1}{2}\,\delta A = 2J$$

et

$$J = \int\int (V - V')(\rho - \rho')^2\,\delta V\,d\rho\,d\rho' = 0.$$

Cela peut s'écrire :

$$J = \int_{\rho_0}^{\rho_1} H\,\delta V\,d\rho = 0$$

en posant :

$$H = V\int_{\rho_0}^{\rho_1} (\rho - \rho')^2\,d\rho' - \int_{\rho_0}^{\rho_1} V'\,(\rho - \rho')^2\,d\rho'.$$

Posons donc pour abréger :

$$\int_{\rho_0}^{\rho_1} d\rho' = \rho_1 - \rho_0;\quad \int_{\rho_0}^{\rho_1} \rho'\,d\rho' = \frac{\rho_1^2 - \rho_0^2}{2};\quad \int_{\rho_0}^{\rho_1} \rho'^2\,d\rho' = \frac{\rho_1^3 - \rho_0^3}{3},$$

$$\int_{\rho_0}^{\rho_1} V'\,d\rho' = \int_{\rho_0}^{\rho_1} V\,d\rho = \alpha;\quad \int_{\rho_0}^{\rho_1} V'\rho'\,d\rho' = \int_{\rho_0}^{\rho_1} V\rho\,d\rho = \beta;\quad \int_{\rho_0}^{\rho_1} V'\rho'^2\,d\rho' = \gamma.$$

Il viendra :

$$H = V\left[\rho^2(\rho_1 - \rho_0) - \rho\,(\rho_1^2 - \rho_0^2) + \frac{\rho_1^3 - \rho_0^3}{3}\right] - \alpha\rho^2 + 2\beta\rho - \gamma.$$

Pour que B soit minimum, il faut d'après les règles du calcul des variations que δB s'annule toutes les fois que J est nul et pour cela il faut que l'on ait :

$$\frac{d^2 V}{d\rho^2} + KH = 0,$$

K étant une constante qu'il reste à déterminer.

Voyons comment le problème pourra être traité.

Posons d'abord

$$\rho = \lambda + \sigma, \quad \lambda = \frac{\rho_1 + \rho_0}{2}, \quad \rho_1 = \lambda + \sigma_0, \quad \rho_0 = \lambda - \sigma_0,$$

nos équations deviendront:

$$\frac{d^2V}{d\sigma^2} + KH = 0,$$

$$H = V\left[2\sigma^2\sigma_0 + \frac{2\sigma_0^3}{3}\right] - \alpha'\sigma^2 + 2\beta'\sigma - \gamma',$$

où:

$$\alpha' = \int V d\sigma, \quad \beta' = \int V\sigma d\sigma, \quad \gamma' = \int V\sigma^2 d\sigma,$$

les intégrales étant prises entre les limites $-\sigma_0$ et $+\sigma_0$.

Posons maintenant $\sigma = \sigma_0 t$,

$$\alpha'' = \int_{-1}^{+1} V dt, \quad \beta'' = \int_{-1}^{+1} Vt dt, \quad \gamma'' = \int_{-1}^{+1} Vt^2 dt. \tag{13}$$

L'équation devient:

$$\frac{1}{K'}\frac{d^2V}{dt^2} + 2V\left[t^2 + \frac{1}{3}\right] - \alpha'' t^2 + 2\beta'' t - \gamma'' = 0. \qquad K' = K\sigma_0^5$$

L'équation (14) contient encore quatre indéterminées, α'', β'', γ'' et K'. Si nous l'intégrons nous trouverons:

$$V = \alpha'' V_1 + \beta'' V_2 + \gamma'' V_3 + \delta'' V_4 + \varepsilon'' V_5,$$

V_1, V_2, V_3, V_4 et V_5 étant des fonctions entièrement connues de t et de K' pendant que δ'' et ε'' sont deux constantes d'intégration.

Pour achever de connaître V il nous restera à déterminer les six constantes α'', β'', γ'', δ'', ε'' et K'. Pour cela nous avons six équations; à savoir, les trois équations (13), l'équation $A = 1$ et les deux relations

$$\frac{dV}{d\rho} = 0 \text{ pour } \rho = \rho_0, \quad \frac{dV}{d\rho} = 0 \text{ pour } \rho = \rho_1.$$

Ces six équations ne suffisent pas toutefois pour déterminer complètement ces six constantes et en particulier K'. On trouve pour K' une infinité de valeurs positives. Nous prendrons la plus petite de ces valeurs que j'appellerai K_0.

Je n'ai pas besoin pour mon objet de calculer effectivement K_0; il me suffit de faire observer que c'est une *constante numérique.*

Il vient alors:

$$K = \frac{K_0}{\sigma_0^5} = \frac{4K_0}{(\rho_1 - \rho_0)^5}.$$

Il nous reste à chercher le minimum de $\frac{B}{A}$ correspondant à cette valeur de K.

Nous trouvons:

$$B = \left[\frac{dV}{d\rho}\,V\right]_{\rho_0}^{\rho_1} - \int \frac{d^2V}{d\rho^2}\,Vd\rho = -\int_{\rho_0}^{\rho_1} \frac{d^2V}{d\rho^2}\,Vd\rho = K\int VHd\rho,$$

ou

$$B = K\int d\rho\, V\left[V\int(\rho-\rho')^2\,d\rho' - \int V'(\rho-\rho')^2\,d\rho'\right]$$
$$= K\int\int V(V-V')(\rho-\rho')^2\,d\rho d\rho'.$$

Or l'intégrale du second membre ne doit pas changer quand on permute ρ et ρ'; on a donc aussi

$$B = -K\int\int V'(V-V')(\rho-\rho')^2\,d\rho d\rho',$$

d'où

$$B = \frac{K}{2}\int\int(V-V')^2(\rho-\rho')^2\,d\rho d\rho' = \frac{K}{2}\,A.$$

Ainsi le minimum de $\frac{B}{A}$ est égal à

$$\frac{K}{2} = \frac{2K_0}{(\rho_1-\rho_0)^5}.$$

Pour une fonction V quelconque on aura donc:

$$\frac{B}{A} > \frac{2K_0}{(\rho_1-\rho\ )^5}. \tag{15}$$

Soit λ la plus grande distance de deux points de la surface du corps solide envisagé on aura

$$\frac{B}{A} > \frac{2K_0}{\lambda^5}.$$

Il est à remarquer que si je n'avais pas voulu indiquer sommairement la manière de calculer la constante numérique K_0, j'aurais pu arriver à la formule (15) en quelques lignes par de simples considérations d'homogénéité.

Il suit de là que l'expression (11) est toujours plus grande que

$$\frac{6K_0W}{\pi\lambda^5}.$$

Par conséquent *pour un solide convexe quelconque on a*:

$$k_2 > \frac{6K_0W}{\pi\lambda^5},$$

K_0 désignant une constante numérique, W le volume du corps, et λ la plus grande distance de deux points de la surface du corps.

Cela posé passons à un solide quelconque; on peut le décomposer en $n-1$ solides partiels convexes. On calculera pour chacun de ces solides le rapport $\frac{W}{\lambda^5}$; imaginons que pour tous ces solides $\frac{W}{\lambda^5}$ soit plus grand que α; on aura pour le solide total

$$k_n > \frac{6K_0}{\pi}\alpha.$$

Or nous pouvons prendre n assez grand et choisir nos $n-1$ solides partiels de telle sorte que la quantité que nous venons d'appeler α soit aussi grande que l'on veut.

Donc k_n sera également aussi grand que l'on voudra.

Donc *pour un solide quelconque k_n croît indéfiniment avec n.*

Nous avons démontré ce théorème dans le cas où $h=0$; cela doit suffire; car k_n est croissant avec h; le théorème peut donc être regardé comme démontré pour toutes les valeurs positives de h.

Je ne veux pas quitter ce sujet sans avoir indiqué un moyen de calculer une limite supérieure de k_n.

Posons $$F = \alpha_1 F_1 + \alpha_2 F_2 + \ldots + \alpha_n F_n,$$

$F_1, F_2, \ldots, F_n$ étant des fonctions données et $\alpha_1, \alpha_2, \ldots, \alpha_n$ des coefficients indéterminés.

Les deux intégrales:

$$B = h\int F^2 d\omega + \int\left[\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2\right] d\tau, \quad A = \int F^2 d\tau,$$

seront des formes quadratiques dépendant des n paramètres $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Formons la forme quadratique:

$$B - \lambda A$$

où λ est un nouveau coefficient indéterminé.

Ecrivons que le discriminant de la forme $B-\lambda A$ est nul; nous obtiendrons une équation algébrique d'ordre n en λ; il est aisé d'établir que cette équation a toutes ces racines réelles (parce que les deux formes B et A sont définies positives; il suffirait d'ailleurs que l'une d'elles le fût. Mais de ce que les deux formes sont toutes deux définies positives, il résulte que les racines sont non-seulement réelles, mais positives).

Soient $$\lambda_1, \lambda_2, \ldots, \lambda_n$$

ces n racines rangées par ordre de grandeur croissante ; je dis qu'on aura :

$$\lambda_1 > k_1,\ \lambda_2 > k_2,\ \lambda_3 > k_3,\ \ldots\ldots,\ \lambda_n > k_n.$$

En effet la plus petite valeur que puisse prendre le rapport $\frac{B}{A}$ quand on fait varier les α doit être plus grande que k_1. Or cette plus petite valeur est λ_1 ; on a donc

$$\lambda_1 > k_1.$$

En vertu de la théorie des formes quadratiques, la forme A peut être décomposée en n carrés et s'écrire :

$$A = \beta_1^2 + \beta_2^2 + \ldots\ldots + \beta_n^2,$$

$\beta_1, \beta_2, \ldots\ldots, \beta_n$ étant des fonctions linéaires de $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$.

De plus cette décomposition peut être faite de telle sorte que

$$B = \lambda_1\beta_1^2 + \lambda_2\beta_2^2 + \ldots\ldots + \lambda_n\beta_n^2.$$

Introduisons maintenant les conditions :

$$\int FU_1 d\tau = \int FU_2 d\tau = \ldots\ldots = \int FU_p d\tau = 0. \tag{16}$$

Le minimum de $\frac{B}{A}$, en tenant compte des conditions (16), devra être plus grand que k_{p+1}.

Cherchons ce minimum. Les conditions (16) sont linéaires par rapport aux β de sorte qu'on peut les écrire :

$$\begin{array}{l} \mu_{11}\beta_1 + \mu_{12}\beta_2 + \ldots\ldots + \mu_{1n}\beta_n = 0, \\ \mu_{21}\beta_1 + \mu_{22}\beta_2 + \ldots\ldots + \mu_{2n}\beta_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mu_{p1}\beta_1 + \mu_{p2}\beta_2 + \ldots\ldots + \mu_{pn}\beta_n = 0. \end{array} \tag{16 bis}$$

Soit λ le minimum cherché du rapport $\frac{B}{A}$. On voit par des calculs qu'il est inutile de reproduire ici que ce minimum qui nous est donné par l'équation suivante, dont je vais expliquer la signification :

$$\Sigma H^2 \Pi(\lambda - \lambda_i) = 0. \tag{17}$$

$\Pi(\lambda - \lambda_i)$ sera le produit de $n - p$ binômes tels que $\lambda - \lambda_i$. Supprimons dans les équations (16 bis) tous les β_i qui ont même indice que les λ_i qui entrent dans le produit $\Pi(\lambda - \lambda_i)$; nous appellerons H le déterminant des équations linéaires ainsi obtenues. Enfin la sommation indiquée par le signe Σ est étendue à toutes les combinaisons des n quantités λ_i prises $n - p$ à $n - p$.

On voit qu'en substituant dans l'équation (17) successivement

$$-\infty,\ \lambda_1,\ \lambda_2,\ \ldots\ldots,\ \lambda_n,\ +\infty$$

on obtient au moins p changements de signe. Par conséquent la plus petite racine de l'équation (17) sera au plus égale à λ_{p+1}. On a donc:

$$\lambda_{p+1} > k_{p+1}. \qquad \text{C. Q. F. D.}$$

Je termine ce paragraphe en donnant une nouvelle manière de calculer une limite supérieure de k_2 pour $h = 0$.

Soient f, g, h trois fonctions quelconques assujetties à la condition suivante: A la surface du corps on aura:

$$\alpha f + \beta g + \gamma h = 0, \tag{18}$$

α, β, γ désignant les trois cosinus directeurs de la normale à cette surface.

De plus nous assujettirons nos trois fonctions à la condition

$$A = \int (f^2 + g^2 + h^2)\, d\tau = 1. \tag{19}$$

L'intégrale

$$B = \int \left[\left(\frac{df}{dx}\right)^2 + \left(\frac{dg}{dy}\right)^2 + \left(\frac{dh}{dz}\right)^2 \right] d\tau$$

aura évidemment un minimum. Cherchons ce minimum. Posons:

$$\theta = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}.$$

Nous trouverons, par le calcul des variations:

$$\alpha\delta f + \beta\delta g + \gamma\delta h = 0 \qquad \text{(à la surface du corps)} \qquad \text{(18 bis)}$$

et

$$\frac{1}{2}\,\delta A = \int (f\delta f + g\delta g + h\delta h)\, d\tau = 0,$$

$$\frac{1}{2}\,\delta B = \int \theta \left(\frac{d\delta f}{dx} + \frac{d\delta g}{dy} + \frac{d\delta h}{dz} \right) d\tau = 0.$$

L'intégration par parties donne:

$$\frac{1}{2}\,\delta B = \int \theta\, (\alpha\delta f + \beta\delta g + \gamma\delta h)\, d\omega - \int \left(\frac{d\theta}{dx}\,\delta f + \frac{d\theta}{dy}\,\delta g + \frac{d\theta}{dz}\,\delta h \right) d\tau,$$

ou en vertu (18 bis)

$$\frac{1}{2}\,\delta B = -\int \left(\frac{d\theta}{dx}\,\delta f + \frac{d\theta}{dy}\,\delta g + \frac{d\theta}{dz}\,\delta h \right) d\tau = 0.$$

Pour que δB soit nul toutes les fois que δA est nul, il faut donc que l'on ait:

$$kf + \frac{d\theta}{dx} = 0, \quad kg + \frac{d\theta}{dy} = 0, \quad kh + \frac{d\theta}{dz} = 0, \tag{20}$$

k étant une constante qu'il reste à déterminer.

Des équations (20) on tire par différentiation et addition:

$$\Delta\theta + k\theta = 0$$

et l'équation (18) devient: $$\frac{d\theta}{dn} = 0.$$

Cela montre que θ est l'une des fonctions U que nous avons définies plus haut; ce ne peut être que U_2; on a $k = k_2$, et pour $\theta = U_2$ on trouve:

$$\frac{B}{A} = \frac{\int \theta^2 d\tau}{\int (f^2 + g^2 + h^2)\, d\tau} = \frac{k_2^2 \int U_2^2 d\tau}{\int \left[\left(\frac{dU_2}{dx}\right)^2 + \left(\frac{dU_2}{dy}\right)^2 + \left(\frac{dU_2}{dz}\right)^2\right] d\tau} = k_2.$$

Pour des fonctions f, g, h quelconques on aura donc

$$\frac{B}{A} > k_2,$$

d'où la règle suivante.

On prend trois fonctions quelconques f, g, h assujetties à la condition (18). (La condition (19) devient inutile dès qu'on considère le rapport $\frac{B}{A}$). Le rapport

$$\frac{\int \left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right)^2 d\tau}{\int (f^2 + g^2 + h^2)\, d\tau}$$

est plus grand que k_2.

§3.—*Lois du Refroidissement.*

Soit V la température d'un point du corps solide; V sera une fonction de x, y, z et de t. Cette température devra satisfaire aux deux équations suivantes:

A l'intérieur du corps: (1) $\dfrac{dV}{dt} = a^2 \Delta V.$

A la surface du corps: (2) $\dfrac{dV}{dn} + hV = 0.$

La température est donnée arbitrairement à l'époque initiale $t=0$. Il peut donc se faire qu'à cette époque $t=0$, l'équation (2) ne soit pas satisfaite; mais elle devra l'être pour toute époque postérieure $t>0$. C'est là une première anomalie qui vaut la peine d'être remarquée.

En voici une seconde: V ne peut pas en général être développée suivant les puissances croissantes de t. Supposons en effet que cela soit possible; qu'arrivera-t-il? Soit V_0 la valeur de V pour $t=0$. On aura pour $t=0$:

$$\frac{dV}{dt}=a^2\Delta V_0, \quad \frac{d^2V}{dt^2}=a^2\Delta\frac{dV}{dt}=a^4\Delta\Delta V_0$$

ou

$$\frac{d^2V}{dt^2}=a^4\Delta^2 V_0$$

en convenant de poser:

$$\Delta^n V=\Delta(\Delta^{n-1}V).$$

On aura ainsi en général:

$$\frac{d^nV}{dt^n}=a^{2n}\Delta^n V_0;$$

de sorte que si le développement était possible, il viendrait:

$$V=V_0+a^2t\Delta V_0+\frac{a^4t^2}{1.2}\Delta^2V_0+\frac{a^6t^3}{1.2.3}\Delta^3V_0+\ldots.$$

Il résulterait de là que la température en un point donné et à un instant donné ne dépendrait plus que de la valeur de V_0 et de ses dérivées en ce point. *La forme du corps solide n'interviendrait en aucune façon.* Cela est absurde.

Pour mieux faire comprendre ces anomalies, nous allons envisager un cas particulier. Imaginons que le solide devienne un mur indéfini compris entre les deux plans

$$x=\pm\pi.$$

Supposons que les deux plans $x=\pm\pi$ qui limitent le mur soient imperméables à la chaleur ce qui revient à supposer $h=0$.

Supposons de plus qu'à l'époque $t=0$; la température initiale V_0 ne dépende que de x et ne change pas quand on change x en $-x$.

Ces propriétés subsisteront évidemment à une époque quelconque. V sera fonction de x et de t seulement et ne changera pas quand on changera x en $-x$.

Dans ces conditions on peut poser:

$$V=\Sigma\phi_m(t)\cos mx.$$

Nous aurons alors
$$\Delta V = \frac{d^2V}{dx^2} = -\Sigma m^2\phi_m(t)\cos mx,$$
$$\frac{dV}{dt} = \Sigma\phi'_m(t)\cos mx.$$

Si dans l'équation
$$\frac{dV}{dt} = a^2\Delta V$$

nous faisons $a^2 = 1$ pour simplifier il vient:
$$\phi'_m(t) + m^2\phi_m(t) = 0$$
d'où
$$\phi_m(t) = A_m e^{-m^2t}$$
et enfin
$$V = \Sigma A_m e^{-m^2t}\cos mx.$$

Donnons-nous la température V_0 à l'instant initial $t = 0$. V_0 pourra toujours être développée par la série de Fourier sous la forme:
$$V_0 = \Sigma A_m \cos mx.$$

La série du 2[d] membre $\Sigma A_m \cos mx$ est toujours convergente, mais la convergence peut n'être pas absolue.

En vertu d'un théorème d'Abel, si l'on prend
$$V = \Sigma A_m e^{-m^2t}\cos mx \tag{3}$$
on aura quand t tendra vers 0
$$\lim V = V_0.$$

L'équation (3) nous fournit donc la solution du problème.

Il semble que la condition $h = 0$ n'ait joué aucun rôle dans ce calcul; ce n'est là qu'une apparence à la quelle il ne faut pas se tromper.

Nous pouvons, il est vrai, dans tous les cas possibles, développer V par la série de Fourier, et écrire:
$$V = \Sigma\phi_m(t)\cos mx.$$

Mais pour que nous ayons le droit d'en conclure
$$\frac{d^2V}{dx^2} = -\Sigma m^2\phi_m(t)\cos mx$$
il faut que la série (et d'ailleurs il suffit)
$$\Sigma m^2\phi_m(t)\cos mx$$
soit convergente.

Pour cela il suffit que l'on puisse trouver un nombre K tel que

$$|m^2\phi_m(t)| < \frac{K}{m^2}$$

ou

$$|\phi_m(t)| < \frac{K}{m^4}. \tag{4}$$

D'après un théorème que j'ai démontré dans le Tome III du Bulletin Astronomique (Sur un moyen d'augmenter la convergence des séries trigonométriques) la condition (4) équivaut à la suivante; la fonction représentée par la série

$$f(x) = \Sigma\phi_m \cos mx$$

devra être continue ainsi que ses trois premières dérivées. Or cette fonction est égale à V entre les limites $-\pi$ et $+\pi$; si on est en dehors de ces limites on a:

$$f(x) = V(x + 2p\pi),$$

p étant un entier positif ou négatif choisi de telle sorte que $x + 2p\pi$ soit compris entre $-\pi$ et $+\pi$.

Comme V est continue ainsi que toutes ses dérivées, il ne peut y avoir de discontinuité qu'aux deux limites $x = \pm\pi$. Si donc nous désignons pour un instant par V', V'', etc. les dérivées successives de V par rapport à x, on devra avoir:

$$V(\pi) = V(-\pi), \tag{5}$$
$$V'(\pi) = V'(-\pi), \tag{6}$$
$$V''(\pi) = V''(-\pi), \tag{7}$$
$$V'''(\pi) = V'''(-\pi). \tag{8}$$

La fonction V étant paire les conditions (5) et (7) sont remplies d'elles-mêmes. D'autre part V' et V''' sont des fonctions impaires de sorte qu'on doit avoir:

$$V'(\pi) = -V'(-\pi), \quad V'''(\pi) = -V'''(-\pi)$$

et que les conditions (6) et (8) peuvent s'écrire:

$$V'(\pi) = V'''(\pi) = 0,$$

c'est-à-dire que pour $x = \pi$ on devra avoir:

$$\frac{dV}{dx} = \frac{d^3V}{dx^3} = 0. \tag{9}$$

Si $h = 0$, on doit avoir pour $x = \pi$:

$$\frac{dV}{dn} = \frac{dV}{dx} = 0.$$

Cette condition ayant lieu, quel que soit t on aura:

$$\frac{d^2V}{dtdx} = \frac{d^3V}{dx^3} = 0.$$

Les conditions (9) sont donc remplies, et notre calcul est légitime, *mais seulement dans le cas de* $h = 0$.

Cela posé considérons la série

$$V_0 = \Sigma A_m \cos mx.$$

En général la série: $\Sigma m^2 A_m \cos mx$

ne sera pas convergente de sorte qu'on ne pourra pas écrire:

$$\Delta V_0 = \frac{d^2V_0}{dx^2} = -\Sigma m^2 A_m \cos mx.$$

Il en résulte qu'on n'aura pas en général:

$$\lim \Delta V = \Delta V_0 \qquad \text{quand } t \text{ tend vers } 0$$

et qu'on n'aura pas non plus:

$$\lim \frac{dV}{dt} = \Delta V_0 \qquad \text{quand } t \text{ tend vers } 0.$$

C'est ce qui explique pourquoi le développement suivant les puissances de t est généralement impossible.

Revenons maintenant au cas général:

On a pour $t > 0$ $\qquad \dfrac{dV}{dn} + hV = 0,$

ou en différentiant par rapport à t:

$$\frac{d^2V}{dtdn} + h\frac{dV}{dt} = 0$$

ou:

$$\frac{d\Delta V}{dn} + h\Delta V = 0.$$

En différentiant p fois on trouverait de même

$$\frac{d\Delta^p V}{dn} + h\Delta^p V = 0.$$

Pour que le développement suivant les puissances de t soit possible, il faut évidemment que l'on ait:

$$\frac{dV_0}{dn} + hV_0 = 0, \quad \frac{d\Delta^p V_0}{dn} + h\Delta^p V_0 = 0. \quad (p = 1, 2, \ldots, \text{ad inf.})$$

Ces conditions, en nombre infini, sont nécessaires ; j'ignore si elles sont suffisantes quoique cela puisse sembler probable. Je n'insiste d'ailleurs sur tous ces points que pour mieux montrer avec quelles précautions il faut toucher aux équations aux dérivées partielles.

Passons maintenant à l'exposé des lois générales du refroidissement.

Considérons d'abord l'intégrale suivante :

$$A = \int V^2 d\tau.$$

Je dis que cette intégrale ira toujours en diminuant ; nous trouvons en effet :

$$\frac{dA}{dt} = 2\int V\frac{dV}{dt}\,d\tau = 2a^2\int V\Delta V d\tau.$$

Il vient ensuite :

$$\int V\Delta V d\tau = \int V\frac{dV}{dn}\,d\omega - \int\left[\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right]d\tau$$

ou en vertu de l'équation (2)

$$\int V\Delta V d\tau = -h\int V^2 d\omega - \int\Sigma\left(\frac{dV}{dx}\right)^2 d\tau = -B < 0$$

et par conséquent : $\frac{dA}{dt} = -2a^2 B < 0.$ C. Q. F. D.

Je dis maintenant que si h n'est pas nul, $\int V^2 d\tau$ tend vers 0 quand t croît indéfiniment. On a en effet $\frac{B}{A} > k_1$

d'où $$\frac{dA}{A dt} < -2a^2 k_1,$$

ou en appelant A_0 la valeur de A pour $t = t_0$,

$$A < A_0 e^{-2a^2 k_1 t}.$$

Si h n'est pas nul, k_1 n'est pas nul non plus, et l'on a :

$$\lim A = 0 \qquad \text{pour } t = \infty$$ C. Q. F. D.

Je dis maintenant que le rapport $\frac{B}{A}$ va constamment en diminuant :

Il vient en effet : $$\frac{d}{dt}\left(\frac{B}{A}\right) = \frac{AdB - BdA}{A^2 dt}$$

de sorte qu'il s'agit de démontrer l'inégalité suivante :

$$A\frac{dB}{dt}-B\frac{dA}{dt}<0.$$

Nous devons donc d'abord calculer $\frac{dB}{dt}$, il viendra :

$$\frac{dB}{dt}=-\frac{d}{dt}\int V\Delta V d\tau=-\int\frac{dV}{dt}\Delta V d\tau-\int V\frac{d\Delta V}{dt}d\tau.$$

L'équation (1) nous permet donc d'écrire :

$$\frac{1dB}{a^2dt}=-\int(\Delta V)^2d\tau-\int V\Delta^2 V d\tau.$$

Or si V et U sont deux fonctions satisfaisant à la surface du corps à l'équation (2) le théorème de Green nous apprend que

$$\int(V\Delta U-U\Delta V)\,d\tau=0.$$

Mais V et ΔV satisfont à l'équation (2). On a donc

$$\int V\Delta^2 V d\tau=\int(\Delta V)^2 d\tau$$

et par conséquent :
$$\frac{dB}{dt}=-2a^2\int(\Delta V)^2 d\tau$$

ce qui montre déjà que B est décroissant.

Nous pouvons écrire (en appelant $d\tau'$ un élément de volume du corps autre que $d\tau$ et désignant par V' la valeur de V au centre de l'élément $d\tau'$)

$$\frac{dA}{dt}=2a^2\int V'\Delta V'd\tau';\quad \frac{dB}{dt}=-2a^2\int(\Delta V')^2 d\tau'$$

et par conséquent :

$$A\frac{dB}{dt}-B\frac{dA}{dt}=-2a^2\left[\int V^2d\tau\int(\Delta V')^2d\tau'-\int V\Delta V d\tau\int V'\Delta V'd\tau'\right]$$

ou
$$A\frac{dB}{dt}-B\frac{dA}{dt}=-2a^2\int\int[(V\Delta V')^2-VV'\Delta V\Delta V']\,d\tau\,d\tau'.$$

Comme rien ne distingue $d\tau$ de $d\tau'$, nous pouvons écrire également :

$$A\frac{dB}{dt}-B\frac{dA}{dt}=-2a^2\int\int[(V'\Delta V)^2-VV'\Delta V\Delta V']\,d\tau\,d\tau',$$

d'où ajoutant les deux valeurs de $A\frac{dB}{dt}-B\frac{dA}{dt}$ et divisant par 2,

$$A\frac{dB}{dt}-B\frac{dA}{dt}=-a^2\int\int[V'\Delta V-V\Delta V']^2\,d\tau\,d\tau'$$

d'où
$$A\frac{dB}{dt}-B\frac{dA}{dt}<0. \qquad \text{C. Q. F. D.}$$

Nous avons vu plus haut que l'on a:

$$A<A_0e^{-2a^2k_1t}$$

cette inégalité peut dans certains cas être remplacée par une autre. Supposons que l'on ait:

$$\int VU_1d\tau=\int VU_2d\tau=\ldots\ldots=\int VU_{n-1}d\tau=0$$

nous aurons d'après définition même des quantités k_n et U_n:

$$\frac{B}{A}>k_n,$$

il viendra donc:
$$\frac{dA}{dt}=-2a^2B<2a^2k_nA$$

et
$$A<A_0e^{-2a^2k_nt}.$$

Etudions maintenant les variations de l'intégrale:

$$J_n=\int VU_nd\tau.$$

Il vient:
$$\frac{dJ_n}{dt}=a^2\int U_n\Delta Vd\tau.$$

Le théorème de Green donne:

$$\int\left(U_n\frac{dV}{dn}-V\frac{dU_n}{dn}\right)d\omega=\int(U_n\Delta V-V\Delta U_n)\,d\tau.$$

En vertu des égalités

$$\frac{dV}{dn}+hV=\frac{dU_n}{dn}+hU_n=0$$

le premier membre est nul; on a donc:

$$\int U_n\Delta Vd\tau=\int V\Delta U_nd\tau=-k_n\int VU_nd\tau=-k_nJ_n$$

d'où
$$\frac{dJ_n}{dt}=-a^2k_nJ_n$$

et
$$J_n=J_n^0e^{-a^2k_nt},$$

J_n^0 représentant la valeur de l'intégrale J_n pour $t=0$.

Etudions encore les variations de l'intégrale

$$H = \int V_1 V_2 d\tau$$

où V_1 représente la température à l'instant t et V_2 la température à l'instant $h - t$. Il vient:

$$\frac{dH}{dt} = a^2 \int (V_2 \Delta V_1 - V_1 \Delta V_2)\, d\tau$$

or en vertu du théorème de Green et des équations

$$\frac{dV_1}{dn} + hV_1 = \frac{dV_2}{dn} + hV_2 = 0$$

le second membre est nul. Donc H est une constante qui ne dépend que de h.

Si nous faisons
$$t = h - t = \frac{h}{2}$$
il vient:
$$V_1 = V_2$$
et
$$H = \int V_1^2 d\tau > 0.$$

Si donc V_1 et V_2 représentent les températures à deux instants quelconques, l'intégrale

$$\int V_1 V_2 d\tau$$

sera toujours positive.

Nous avons vu plus haut que l'intégrale

$$\int V^2 d\tau$$

considérée comme fonction du temps va toujours en décroissant.

Donc H qui si l'on fait $t = \frac{h}{2}$ se réduit à

$$\int V_1^2 d\tau$$

sera une fonction décroissante de h.

Or nous trouvons: $$\frac{dH}{dh} = a^2 \int V_1 \Delta V_2 d\tau < 0.$$

Nous trouvons de même

$$\int V_2 \Delta V_1 d\tau = \int V_1 \Delta V_2 d\tau < 0.$$

Si donc V_1 et V_2 représentent les températures à deux instants quelconques, l'intégrale

$$\int V_1 \Delta V_2 d\tau$$

sera toujours négative.

Arrivons maintenant au problème principal; étant donnée la température au temps $t = 0$, trouver la température à un instant quelconque.

Soit V_0 la température à l'instant $t = 0$.

La solution classique consisterait en ceci:

Développer V_0 en une série de la forme suivante:

$$V_0 = A_1 U_1 + A_2 U_2 + \dots + A_n U_n + \dots,$$

les A étant des constantes.

On aura ensuite à un instant quelconque:

$$V = A_1 e^{-a^2 k_1 t} U_1 + A_2 e^{-a^2 k_2 t} U_2 + \dots + A_n e^{-a^2 k_n t} U_n + \dots$$

Cette solution est subordonnée à la possibilité du développement, et c'est cette possibilité que nous ne sommes pas encore en mesure de démontrer d'une manière générale.

Voici toutefois ce que nous pouvons dire.

Soient $A_1, A_2, \dots, A_n$ des coefficients quelconques; posons:

$$V_0 = A_1 U_1 + A_2 U_2 + \dots + A_n U_n + R_0$$

et proposons-nous de déterminer les coefficients A de telle façon que l'erreur moyenne commise soit minimum.

Nous prendrons, à l'exemple de M. Tchebicheff, pour mesure de l'erreur moyenne commise l'intégrale suivante:

$$S_0 = \int R_0^2 d\tau.$$

Cherchons donc le minimum de l'intégrale

$$\int (V_0 - A_1 U_1 - A_2 U_2 - \dots - A_n U_n)^2 d\tau.$$

Cette intégrale sera minimum quand on aura:

$$\int U_p (V_0 - A_p U_p) d\tau = 0$$

ou (puisque nous avons par définition

$$\int U_p^2 d\tau = 1, \quad \int V_0 U_p d\tau = J_p^0)$$

quand on aura: $$A_p = J_p^0.$$

Nous sommes donc conduits à écrire:

$$V_0 = J_1^0 U_1 + J_2^0 U_2 + \ldots . + J_n^0 U_n + R_0.$$

Il résulte de là que l'erreur moyenne commise S_0 va en diminuant quand n augmente, mais non que cette erreur moyenne tende vers 0 quand n croît au delà de toute limite. D'ailleurs S_0 pourrait tendre vers 0 sans que R_0 tendît vers 0.

Remplaçons toutefois V_0 par sa valeur approchée

$$\sum_{p=1}^{p=n} J_p^0 U_p.$$

Nous en déduirons

$$V = \sum_{p=1}^{p=n} J_p^0 e^{-a^2 k_p t} U_p = \sum J_p U_p,$$

nous rappelons que nous avons posé

$$\int V U_p d\tau = J_p = J_p^0 e^{-a^2 k_p t}.$$

Posons donc: $$V = J_1 U_1 + J_2 U_2 + \ldots . + J_n U_n + R,$$

et prenons pour mesure de l'erreur moyenne commise:

$$S = \int R^2 d\tau,$$

je me propose de démontrer que l'on peut prendre n assez grand pour que l'erreur moyenne S commise sur la température à un instant donné soit aussi petite qu'on le veut.

En effet R satisfait aux équations

$$\frac{dR}{dt} = a^2 \Delta R, \quad \frac{dR}{dn} + hR = 0.$$

Si donc la température initiale était R_0, la température à un instant ultérieur serait représentée par R.

De plus on a, comme il est aisé de le vérifier:

$$\int R U_1 d\tau = \int R U_2 d\tau = \ldots . = \int R U_n d\tau = 0 .$$

Donc d'après un lemme que nous avons démontré plus haut, on aura:

$$S < S_0 e^{-a^2 k_{n+1} t}.$$

Or quand n croît au delà de toute limite, S_0 décroît (sans que nous sachions s'il tend vers 0), k_{n+1} croît au delà de toute limite et l'exponentielle $e^{-a^2 k_{n+1} t}$ tend vers 0. Donc S tend vers 0. C. Q. F. D.

Nous avons donc démontré que l'erreur moyenne S tend vers 0, mais non que R tend vers 0. Cela peut toutefois nous suffire pour le moment. En effet comment pourrait-il arriver que S tendît vers 0 sans qu'il en fût de même de R? Il faudrait pour cela que la valeur de R subît des oscillations d'autant plus rapides que n serait plus grand, de telle façon que pour n très grand, R prendrait en des points très rapprochés des valeurs très différentes. Aucun physicien ne doutera que si un pareil état de choses existait à l'instant initial, il ne saurait subsister. C'est ce qui m'engage à me contenter pour le moment des considérations qui précèdent.

Je terminerai ce paragraphe par la remarque suivante:

Si V_0 est partout positif, V sera aussi positif pour toutes les valeurs de t et pour tous les points du corps. Or quand t croît indéfiniment, le rapport:

$$\frac{V}{J_1 U_1}$$

tend vers l'unité.

Donc U_1 doit être une fonction qui est positive en tous les points du corps.

L'égalité

$$\int U_1 U_n d\tau = 0 \qquad (n > 1)$$

montre que la fonction U_1 est la seule des fonctions U_n qui jouisse de cette propriété.

§4.—*Propriétés des Fonctions U_n.*

Reprenons la fonction U_n définie par les équations:

$$\Delta U_n + k_n U_n = 0, \quad \frac{dU_n}{dn} + h U_n = 0, \quad \int U_n^2 d\tau = 1.$$

Ou en supprimant les indices qui nous sont inutiles pour le moment:

$$\Delta U + kU = 0, \quad \frac{dU}{dn} + hU = 0, \quad \int U^2 d\tau = 1.$$

Soit T une fonction satisfaisant comme U à l'équation:

$$\Delta T + kT = 0.$$

Si T est finie et continue ainsi que ses dérivées à l'intérieur du corps, on aura

$$\int \left(U\frac{dT}{dn} - T\frac{dU}{dn} \right) d\omega = 0$$

ou

$$\int U\left(\frac{dT}{dn} + hT\right) d\omega = 0,$$

les intégrales étant étendues à la surface du corps.

Supposons maintenant que la fonction T ne soit plus finie à l'intérieur du corps qu'elle devienne infinie au point (x_0, y_0, z_0) situé à l'intérieur du corps; mais de telle façon que la différence $T - \frac{1}{r}$ (où r désigne la distance des deux points x, y, z et x_0, y_0, z_0) reste finie ainsi que ses dérivées.

On aura alors: $$\int \left(U\frac{dT}{dn} - T\frac{dU}{dn} \right) d\omega = -4\pi U^0,$$

U^0 désignant la valeur de U au point x_0, y_0, z_0. C'est ce que j'ai déjà exposé à propos de la Diffraction dans ma Théorie Mathématique de la Lumière.

In vient donc: $$\int U\left(\frac{dT}{dn} + hT\right) d\omega = -4\pi U^0.$$

Soit maintenant: $$\alpha = \sqrt{k}$$

et $$T = \frac{e^{i\alpha r}}{r},$$

r désignant encore la distance du point mobile x, y, z au point fixe x_0, y_0, z_0.

On aura alors: $$\int U\left(\frac{dT}{dn} + hT\right) d\omega = 0 \text{ ou } -4\pi U^0,$$

selon que le point x_0, y_0, z_0 est extérieur ou intérieur au corps.

On a d'ailleurs dans ce cas:

$$\frac{dT}{dn} = \cos\psi \frac{e^{i\alpha r}}{r}\left(i\alpha - \frac{1}{r}\right),$$

ψ étant l'angle de la normale à la surface du corps au point x, y, z avec la droite qui joint ce point au point fixe x_0, y_0, z_0.

Nous poserons pour abréger

$$H = \frac{dT}{dn} + hT$$

et nous regarderons H soit comme fonction de r et de

$$r \cos \psi = \xi$$

soit comme fonction de x, y, z et de x_0, y_0, z_0.

On aura alors:

$$r \cos \psi = \lambda (x_0 - x) + \mu (y_0 - y) + \nu (z_0 - z),$$

λ, μ, ν désignant les trois cosinus directeurs de la normale à la surface du corps. On en déduit:

$$\frac{dH}{dx_0} = \frac{dH}{dr} \frac{x_0 - x}{r} + \frac{dH}{d\xi} \lambda.$$

On a ensuite si le point x_0, y_0, z_0 est intérieur au corps:

$$U_0 = \frac{-1}{4\pi} \int HU d\omega, \quad \frac{dU_0}{dx_0} = \frac{-1}{4\pi} \int \frac{dH}{dx_0} U d\omega.$$

Cela va nous permettre de trouver une limite supérieure de U_0 et de ses dérivées.

Soit en effet A la plus grande valeur que puisse prendre $|U|$ à la surface du corps, on aura évidemment:

$$|U_0| < \frac{A}{4\pi} \int |H| d\omega, \quad \left| \frac{dU_0}{dx_0} \right| < \frac{A}{4\pi} \int \left| \frac{dH}{dx_0} \right| d\omega.$$

Les deux intégrales qui entrent dans ces deux inégalités

$$\int |H| d\omega \text{ et } \int \left| \frac{dH}{dx_0} \right| d\omega$$

se calculent aisément quand on connaît la forme de la surface du corps et le nombre positif k. Elles ne dépendent que de x_0, y_0, z_0.

Quant au coefficient A, nous n'avons jusqu'à présent aucun moyen de le déterminer.

Commençons par étudier les variations de la première intégrale $\int |H| d\omega$.

Cette intégrale est évidemment finie tant que le point x_0, y_0, z_0 reste intérieur au corps. On a en effet:

$$|H| < \frac{\alpha}{r} + \frac{1}{r^2} + \frac{h}{r}$$

de sorte qu'il viendra :

$$\int |H| d\omega < S\left(\frac{\alpha}{\rho} + \frac{1}{\rho^2} + \frac{h}{\rho}\right),$$

S désignant la surface totale du corps et ρ la plus courte distance du point x_0, y_0, z_0 à cette surface. Je dis maintenant que notre intégrale tendra encore vers une limite finie quand le point x_0, y_0, z_0 se rapprochera indéfiniment de cette surface.

Posons en effet :
$$H = \frac{h}{r} - \frac{\cos\psi}{r^2} + H_1.$$

H_1 sera une fonction qui ne deviendra pas infinie même quand r s'annulera. Nous trouvons en effet :

$$H_1 = \frac{\cos\psi}{r^2}\left[iare^{iar} - e^{iar} + 1\right] + \frac{h}{r}\left(e^{iar} - 1\right),$$

$$\left|\frac{e^{iar} - 1}{r}\right| = \frac{2\left|\sin\frac{ar}{2}\right|}{r} < \alpha; \quad \left|\frac{iare^{iar} - e^{iar} + 1}{r^2}\right| < 4\alpha^2$$

et enfin :
$$|H_1| < h\alpha + 4\alpha^2,$$

d'où :
$$\int |H| d\omega < h\alpha S + 4\alpha^2 S + h\int \frac{d\omega}{r} + \int \frac{d\omega}{r^2}|\cos\psi|.$$

Il est aisé de voir que quand le point x_0, y_0, z_0 se rapproche indéfiniment de la surface du corps, les intégrales $\int \frac{d\omega\,|\cos\psi|}{r^2}$ et l'intégrale $\int \frac{d\omega}{r}$ tendent vers des limites finies.

Si d'abord le corps est convexe, de telle façon qu'une droite ne puisse couper sa surface en plus de deux points; $\cos\psi$ est positif et l'on a :

$$\int \frac{|\cos\psi|}{r^2} d\omega = \int \frac{\cos\psi}{r^2} d\omega = 4\pi.$$

Si le corps n'est pas convexe, et qu'une droite puisse rencontrer sa surface en n points on aura :

$$\int \frac{|\cos\psi|}{r^2} d\omega < 4(n-1)\pi$$

car l'intégrale s'obtient en décrivant du point x_0, y_0, z_0 comme centre une sphère de rayon 1 et en faisant la perspective de la surface du corps sur la surface de cette sphère, le centre de la dite sphère étant pris comme centre de la perspective. Un point de la sphère sera alors au plus la perspective de $n-1$ points de la

surface du corps; de sorte que la somme *arithmétique* des aires des perspectives des divers éléments de cette surface sera au plus égale à $n-1$ fois la surface de la sphère. Or cette somme arithmétique est précisément l'intégrale, $\int \frac{|\cos\psi|}{r^2}\,d\omega$. La somme algébrique serait l'intégrale $\int \frac{\cos\psi}{r^2}\,d\omega$.

Ainsi $\int \frac{|\cos\psi|}{r^2}\,d\omega$ tend vers une limite finie; il reste à démontrer qu'il en est de même de

$$\int \frac{d\omega}{r}.$$

Or cette intégrale représente le potentiel d'une couche uniforme de matière répandue à la surface du corps, et l'on sait que ce potentiel est fini.

Si nous passons maintenant à l'inégalité:

$$\frac{dU_0}{dx_0} < A\int \left|\frac{dH}{dx_0}\right| d\omega,$$

elle nous permet de démontrer qu'à l'intérieur du corps les dérivées premières (et on le démontrerait de la même façon pour les dérivées d'ordre supérieure) de la fonction U_0 restent finies; mais elle ne nous permet pas de voir si ces dérivées tendent vers une limite finie quand le point x_0, y_0, z_0 se rapproche indéfiniment de la surface du corps.

Nous allons maintenant chercher à obtenir une limite supérieure du coefficient A.

Pour cela il nous faut démontrer que U est une fonction continue.

Cela est évident pour les points situés à l'intérieur du corps puisque nous venons de voir qu'en ces points on peut trouver une limite supérieure des dérivées de U.

Il reste à démontrer que U est encore une fonction continue sur la surface du corps, et pour cela il nous faut une expression de U_0 quand le point x_0, y_0, z_0 est sur cette surface.

Nous avons trouvé plus haut

$$\int UHd\omega = 0 \text{ ou } -4\pi U_0$$

selon que le point x_0, y_0, z_0 est extérieur ou intérieur du corps.

En vertu des mêmes principes, on aura

$$\int UH d\omega = -2\pi U_0$$

si le point x_0, y_0, z_0 est sur la surface même du corps.

Si x_0', y_0', z_0' est un autre point situé également sur la surface du corps et si H' et U_0' sont deux fonctions formées avec le point x_0', y_0', z_0' comme H et U_0 le sont avec le point x_0, y_0, z_0; il viendra:

$$\int U(H - H')\, d\omega = 2\pi (U_0' - U_0)$$

d'où:

$$|U_0' - U_0| < A \int |H - H'|\, d\omega.$$

Il nous reste à établir que: $\int |H - H'|\, d\omega$

tend vers 0 quand le point x_0', y_0', z_0' se rapproche indéfiniment du point x_0, y_0, z_0.

Remarquons d'abord que nous pouvons poser:

$$H = \frac{h}{r} - \frac{\cos\psi}{r^2} + H_1,$$

$$H' = \frac{h}{r'} - \frac{\cos\psi'}{r'^2} + H_1',$$

H_1 a même signification que plus haut; H_1' est formé avec le point x_0', y_0', z_0' comme H_1 avec le point x_0, y_0, z_0; r' désigne la distance du point x_0', y_0', z_0' au point x, y, z et l'angle ψ' est défini avec le point x_0', y_0', z_0' comme l'angle ψ avec le point x_0, y_0, z_0.

On a donc:

$$\int |H - H'|\, d\omega < h \int \left| \frac{1}{r} - \frac{1}{r'} \right| d\omega + \int \left| \frac{\cos\psi}{r^2} - \frac{\cos\psi'}{r'^2} \right| d\omega + \int |H_1 - H_1'|\, d\omega.$$

Il suffit donc de démontrer que les trois intégrales

$$\int \left| \frac{1}{r} - \frac{1}{r'} \right| d\omega, \quad \int \left| \frac{\cos\psi}{r^2} - \frac{\cos\psi'}{r'^2} \right| d\omega, \quad \int |H_1 - H_1'|\, d\omega$$

tendent vers 0 quand les deux points se rapprochent indéfiniment. Cela est évident pour la troisième; car H_1 est une fonction continue et finie de x_0, y_0 et z_0.

Démontrons-le maintenant pour la première.

Décomposons la surface S du corps en deux régions σ et σ'; et supposons que les deux points x_0, y_0, z_0 et x_0', y_0', z_0' appartiennent à la région σ.

Je dis que je pourrai prendre ces deux points assez rapprochés pour que

$$\int_S \left| \frac{1}{r} - \frac{1}{r'} \right| d\omega = \int_\sigma \left| \frac{1}{r} - \frac{1}{r'} \right| d\omega + \int_{\sigma'} \left| \frac{1}{r} - \frac{1}{r'} \right| d\omega < \varepsilon.$$

Tout d'abord nous savons que les deux intégrales

$$\int \frac{d\omega}{r}, \quad \int \frac{d\omega}{r'}$$

sont finies et déterminées. Il en résulte que nous pourrons prendre la région σ assez petite pour que :

$$\int_\sigma \frac{d\omega}{r} < \frac{\varepsilon}{3}, \quad \int_\sigma \frac{d\omega}{r'} < \frac{\varepsilon}{3}$$

quelle que soit la position du point x_0', y_0', z_0' dans cette région σ d'où :

$$\int_\sigma \left| \frac{1}{r} - \frac{1}{r'} \right| d\omega < \frac{2\varepsilon}{3}.$$

La région σ doit désormais être regardée comme entièrement déterminée, mais nous pouvons encore faire varier dans cette région le point x_0', y_0', z_0'.

Si maintenant r' désigne la distance du point x_0', y_0', z_0' à un point x, y, z quelconque de la région σ'; $\frac{1}{r'}$ sera une fonction finie et continue de x_0', y_0', z_0' et cette fonction tendra *uniformément* vers $\frac{1}{r}$ quand x_0', y_0', z_0' tendront vers x_0, y_0, z_0. Cela sera vrai tant que le point x, y, z restera sur la région σ', puisque les points x_0, y_0, z_0 et x_0', y_0', z_0' n'appartiennent pas à cette région.

On pourra donc prendre le point x_0', y_0', z_0' assez voisin du point x_0, y_0, z_0 pour que

$$\int_{\sigma'} \left| \frac{1}{r} - \frac{1}{r'} \right| d\omega < \frac{\varepsilon}{3}$$

et par conséquent :

$$\int_S \left| \frac{1}{r} - \frac{1}{r'} \right| d\omega < \varepsilon.$$

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On établirait de la même manière que

$$\lim \int \left| \frac{\cos \psi}{r^2} - \frac{\cos \psi'}{r'^2} \right| d\omega < 0.$$

On a donc :

$$\lim \int | H - H' | \, d\omega = 0 \qquad \lim | U_0 - U_0' | = 0$$

ce qui signifie que la fonction U est continue à la surface du corps.

Cette démonstration ne montre toutefois qu'une chose, c'est que U_0' tend vers U_0, quand le point x_0, y_0, z_0 appartient à la surface du corps et que le point x_0', y_0', z_0' s'en rapproche en restant lui-même sur la surface du corps. Elle ne nous apprend rien sur ce qui arrive quand le point x_0', y_0', z_0' est intérieur au corps et se rapproche de x_0, y_0, z_0 soit normalement, soit obliquement à la surface du corps. On pourrait observer toutefois que

$$\frac{dU}{dn} = -hU$$

étant fini, U_0' doit tendre vers U_0 quand la droite qui joint les deux points est normale à la surface et il serait aisé d'en conclure, par un petit raisonnement très simple, qu'il en est encore de même quand cette droite est oblique.

Mais cela ne saurait nous suffire, parce que nous avons besoin pour notre objet d'assigner une limite supérieure de $|U_0' - U_0|$.

Si comme nous le supposons le point x_0, y_0, z_0 est sur la surface du corps et le point x_0', y_0', z_0' à l'intérieur, on aura:

$$\int HUd\omega - 2\pi U_0 = -4\pi U_0,$$

$$\int H' Ud\omega = -4\pi U_0'$$

d'où

$$\int (H - H')\, Ud\omega - 2\pi U_0 = -4\pi (U_0 - U_0').$$

Nous avons donc:

$$-4\pi (U_0 - U_0') = h\int \left(\frac{1}{r} - \frac{1}{r'}\right) Ud\omega$$
$$+\int (H_1 - H_1')\, Ud\omega + \int \left(\frac{\cos\psi'}{r'^2} - \frac{\cos\psi}{r^2}\right) Ud\omega - 2\pi U_0.$$

Il nous faut démontrer que les trois intégrales du 2[d] membre tendent respectivement vers 0, 0 et $2\pi U_0$ quand le point x_0', y_0', z_0' se rapproche indéfiniment du point x_0, y_0, z_0.

Or on a:

$$\left|\int (H_1 - H_1')\, Ud\omega\right| < A\int |H_1 - H_1'|\, d\omega$$

et on verrait comme plus haut que

$$\lim \int |H_1 - H_1'|\, d\omega = 0.$$

Il n'y aurait rien à changer à la démonstration que nous avons donnée dans le cas qui précède.

De même on a :

$$\left| \int \left(\frac{1}{r} - \frac{1}{r'} \right) U d\omega \right| < A \int \left| \frac{1}{r} - \frac{1}{r'} \right| d\omega$$

et

$$\lim \int \left| \frac{1}{r} - \frac{1}{r'} \right| d\omega = 0.$$

Ici encore il n'y a rien à changer à la démonstration que nous avons donnée dans le cas précédent.

Il reste à établir que :

$$\lim \int \left(\frac{\cos \psi'}{r'^2} - \frac{\cos \psi}{r^2} \right) U d\omega = 2\pi U_0.$$

Décomposons la surface S du corps en deux régions σ et σ' et supposons encore que le point x_0, y_0, z_0 soit situé dans la région σ ; il viendra :

$$\int \left(\frac{\cos \psi'}{r'^2} - \frac{\cos \psi}{r^2} \right) U d\omega = \int_{\sigma'} \left(\frac{\cos \psi'}{r'^2} - \frac{\cos \psi}{r^2} \right) U d\omega$$

$$+ \int_{\sigma} \left(\frac{\cos \psi'}{r'^2} - \frac{\cos \psi}{r^2} \right) (U - U_0) d\omega + \int_{\sigma} \left(\frac{\cos \psi'}{r'^2} - \frac{\cos \psi}{r^2} \right) U d\omega. \qquad (2)$$

Je dis que je puis prendre le point x'_0, y'_0, z'_0 assez voisin de x_0, y_0, z_0 pour que la différence de $2\pi U_0$ et du premier membre de l'identité (2) soit plus petite en valeur absolue que ε.

En premier lieu, la fonction U étant continue à la surface du corps, nous pourrons prendre la région σ assez petite pour que sur cette région on ait :

$$| U - U_0 | < \zeta,$$

ζ étant une quantité aussi petite que l'on veut. Il vient alors :

$$\left| \int_{\sigma} \left(\frac{\cos \psi'}{r'^2} - \frac{\cos \psi}{r^2} \right) (U - U_0) d\omega \right| < \zeta \int_{\sigma} \left| \frac{\cos \psi'}{r'^2} - \frac{\cos \psi}{r^2} \right| d\omega.$$

Or nous venons de trouver

$$\int \left| \frac{\cos \psi}{r^2} \right| d\omega < 4(n-1)\pi,$$

n étant le nombre maximum des intersections d'une droite avec la surface du corps.

Il vient donc:

$$\left|\int_\sigma\left(\frac{\cos\psi'}{r'^2}-\frac{\cos\psi}{r^2}\right)(U-U)\,d\omega\right| < 8\,(n-1)\,\pi\zeta < \frac{\varepsilon}{3}.$$

Car, ζ étant aussi petit que je veux, je puis toujours prendre

$$\zeta < \frac{\varepsilon}{24\,(n-1)\,\pi}.$$

La région σ doit être regardée maintenant comme entièrement déterminée, mais je puis encore faire varier le point x_0', y_0', z_0'.

Nous voyons d'abord que le point x_0, y_0, z_0 ne faisant pas partie de la région σ', on a:

$$\lim\int_{\sigma'}\left|\frac{\cos\psi'}{r'^2}-\frac{\cos\psi}{r^2}\right|d\omega = 0,$$

de sorte qu'on peut prendre le point x_0', y_0', z_0' assez voisin du point x_0, y_0, z_0 pour que:

$$\int_{\sigma'}\left|\frac{\cos\psi'}{r'^2}-\frac{\cos\psi}{r^2}\right|d\omega < \frac{\varepsilon}{3A}$$

et par conséquent pour que:

$$\left|\int_{\sigma'}\left(\frac{\cos\psi'}{r'^2}-\frac{\cos\psi}{r^2}\right)d\omega\right| < \frac{\varepsilon}{3}.$$

D'autre part l'intégrale

$$\int_\sigma\frac{\cos\psi'}{r'^2}\,d\omega$$

réprésente l'angle solide sous lequel on voit le contour de la région σ du point x_0', y_0', z_0', et l'intégrale

$$\int_\sigma\frac{\cos\psi}{r^2}\,d\omega$$

réprésente l'angle solide sous lequel on voit le même contour du point x_0, y_0, z_0. Il en résulte que:

$$\lim\int_\sigma\frac{\cos\psi'}{r'^2}\,d\omega = \int_\sigma\frac{\cos\psi}{r^2}\,d\omega + 2\pi.$$

On peut donc prendre le point x_0', y_0', z_0' assez voisin de x_0, y_0, z_0 pour que

$$\left|\int_\sigma\left(\frac{\cos\psi'}{r'^2}-\frac{\cos\psi}{r^2}\right)d\omega \quad - 2\pi\right| < \frac{\varepsilon}{3A}$$

ou:
$$\left|\int_\sigma\left(\frac{\cos\psi'}{r'^2}-\frac{\cos\psi}{r^2}\right)U_0 d\omega-2\pi U_0\right|<\frac{\varepsilon}{3}.$$

Il vient alors
$$\left|\int_S\left(\frac{\cos\psi'}{r'^2}-\frac{\cos\psi}{r^2}\right)U d\omega-2\pi U_0\right|<\varepsilon. \qquad \text{C. Q. F. D.}$$

Voici le résumé de cette longue discussion.

Soient x_0, y_0, z_0; x'_0, y'_0, z'_0 deux points situés soit à l'intérieur du corps, soit à sa surface. Soient U_0 et U'_0 les valeurs de U en ces deux points; on aura:

$$|U_0-U'_0|<A\theta,$$

A étant la plus grande valeur de $|U|$ à la surface du corps; quant à θ ce sera une fonction de $x_0, y_0, z_0, x'_0, y'_0, z'_0$ que l'on pourra déterminer complètement à l'aide des considérations qui précèdent quand on connaîtra la forme du corps. Ces considérations nous fournissent en effet une limite supérieure de $|U_0-U'_0|$ puisqu'elles nous montrent comment on doit choisir les deux points x_0, y_0, z_0, x'_0, y'_0, z'_0 pour que $|U_0-U'_0|$ soit plus petit que ε.

Je n'attirerai l'attention que sur deux des propriétés de la fonction θ. Elle est essentiellement positive et elle tend uniformément vers 0 quand le point x'_0, y'_0, z'_0 se rapproche indéfiniment du point x_0, y_0, z_0.

Regardons d'abord le point x_0, y_0, z_0 comme fixe et situé sur la surface du corps et faisons varier le point x'_0, y'_0, z'_0. Nous pourrons diviser le volume du corps en deux régions que nous appellerons R et R' et que nous définirons comme il suit:

Quand le point x'_0, y'_0, z'_0 sera dans la région R on aura:

$$\theta<1.$$

Quand ce point sera dans la région R' on aura:

$$\theta>1.$$

La région R existe certainement et son volume ne peut être nul, puisque θ est très voisin de 0 quand le point x'_0, y'_0, z'_0 est très voisin de x_0, y_0, z_0.

Si nous supposons en particulier que le point x_0, y_0, z_0 soit celui des points de la surface du corps où $|U|$ atteint sa valeur maximum A; on aura:

$$|U'_0|>A(1-\theta)$$

tant que le point x'_0, y'_0, z'_0 restera à l'intérieur de la région R.

Soit $d\tau'$ l'élément de volume du corps dont le centre de gravité est x_0', y_0', z_0'.

On aura:
$$\int U_0'^2 d\tau' = 1,$$
l'intégrale étant étendue au corps tout entier; et par conséquent
$$\int_R U_0'^2 d\tau' < 1,$$
l'intégrale étant étendue seulement à la région R. On en déduit:
$$A^2 \int_R (1-\theta)^2 d\tau' < 1.$$
Cette inégalité est vraie pourvu que l'on ait choisi pour le point x_0, y_0, z_0 celui des points de la surface du corps pour lequel
$$|U| = A.$$
Malheureusement nous ne savons pas quel est celui des points de la surface pour lequel cela a lieu. Mais nous pouvons tourner la difficulté de la façon suivante. L'intégrale
$$\int_R (1-\theta)^2 d\tau'$$
peut être calculée dès que l'on connaît la forme du corps et le point x_0, y_0, z_0. C'est donc une fonction de x_0, y_0, z_0. Cette fonction ne peut jamais s'annuler. Elle aura donc un minimum M que l'on pourra déterminer dès qu'on connaîtra la forme du corps. Il vient ainsi
$$A^2 M < 1$$
d'où
$$A < \frac{1}{\sqrt{M}}.$$
Ainsi nous pouvons déterminer une limite supérieure du coefficient A et par conséquent une limite supérieure des dérivées d'ordre quelconque de U en un point quelconque de l'intérieur du corps.

§5.—*Retour à l'Hypothèse moléculaire.*

Dans les raisonnements qui remplissent les trois paragraphes précédents, il y a un point faible que j'ai déjà signalé plus haut.

Après avoir montré qu'une certaine intégrale ne pouvait pas s'annuler, nous

en avons conclu que cette intégrale devait avoir un minimum, et nous avons déterminé la fonction U qui correspond à ce minimum par le calcul des variations. Or cette application n'eût été légitime que si nous avions démontré d'avance la continuité de cette fonction U. C'est d'ailleurs la même objection qui empêche de regarder comme rigoureuse la démonstration du principe de Dirichlet par Riemann.

Il est vrai que dans le § précédent, nous avons trouvé une limite supérieure de la dérivée de cette fonction U; mais, si l'on voulait s'en servir pour justifier l'emploi du calcul des variations, on commettrait une pétition de principe; tout au plus ce résultat peut-il mettre sur la voie dans la recherche d'une démonstration satisfaisante.

Il faudrait donc, pour obtenir une théorie analytiquement rigoureuse, employer des procédés analogues à ceux qui permettent d'établir le principe de Dirichlet et peut être des procédés plus compliqués encore.

Je ne l'ai pas fait; mais j'ai pensé qu'il était possible d'obtenir une démonstration rigoureuse au point de vue physique de la façon suivante. Au lieu de considérer l'équation différentielle de Fourier en elle-même, rappelons-nous quelle est sa signification physique et comment on l'a obtenue.

On considère un corps solide formé d'un très grand nombre de molécules. Soient

$$M_1, M_2, \ldots, M_n$$

ces molécules, n est un très grand nombre. Soient

$$V_1, V_2, \ldots, V_n$$

les températures de ces molécules.

La molécule M_i enverra à la molécule M_k une quantité de chaleur égale à

$$C_{ik}(V_i - V_k),$$

C_{ik} étant un coefficient indépendant des températures, ne dépendant que de la distance des deux molécules; ce coefficient est très petit dès que cette distance devient sensible.

En outre, cette molécule M_i rayonnera au dehors une quantité de chaleur égale à:

$$C_i V_i,$$

C_i étant un coefficient qui n'est sensible que pour les molécules superficielles.

Si nous choisissons l'unité de chaleur de façon que chacune de nos molécules, que nous supposons toutes pareilles entre elles ait pour chaleur spécifique l'unité, nous pourrons écrire :

$$\frac{dV_i}{dt} + \sum_{k=1}^{k=n} C_{ik}(V_i - V_k) + C_i V_i = 0 \tag{1}$$

et nous aurons n équations pareilles en faisant $i = 1, 2, \ldots, n$.

C'est en transformant le système (1) que Fourier est arrivé aux équations qui nous ont occupés dans les §§ précédents. Pour cela il passe à la limite, de façon à passer des équations aux différences finies aux équations différentielles et en tenant compte de l'isotropie du corps. Mais si l'étude de ces équations différentielles nous conduit à une de ces difficultés qui tiennent à la considération de l'infini, nous ne devons pas oublier que cette difficulté est factice, puisque au point de vue purement physique, ces équations différentielles ne sont là que pour remplacer des équations aux différences finies qui en diffèrent très peu et pour lesquelles cette difficulté n'existe pas. Il y a donc intérêt à étudier le système (1) en lui-même.

Cette étude ne présente aucune difficulté puisqu'il s'agit d'un système d'équations linéaires à coefficients constants.

Posons donc : $$V_i = U_i e^{-\alpha t},$$

les U_i et α étant des constantes ; les équations (1) deviendront :

$$\alpha U_i = \Sigma_k C_{ik}(U_i - U_k) + C_i U_i. \tag{2}$$

En éliminant les n constantes U_i entre ces n équations (2), on arrive à une équation de degré n en α que j'écrirai :

$$F(\alpha) = 0. \tag{3}$$

Soient $\alpha_1, \alpha_2, \ldots, \alpha_n$ les n racines de l'équation (3); considérons une de ces racines que j'appelle α_p. Quand dans les équations (2) on fera $\alpha = \alpha_p$, ces équations deviendront compatibles et on en tirera :

$$U_1 = U_{p1}, \ U_2 = U_{p2}, \ldots, \ U_n = U_{pn}.$$

L'intégrale générale des équations (1) sera alors:

$$\begin{aligned} V_1 &= A_1 e^{-a_1 t} U_{11} + A_2 e^{-a_2 t} U_{21} + \dots + A_n e^{-a_n t} U_{n1}, \\ V_2 &= A_1 e^{-a_1 t} U_{12} + A_2 e^{-a_2 t} U_{22} + \dots + A_n e^{-a_n t} U_{n2}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ V_n &= A_1 e^{-a_1 t} U_{1n} + A_2 e^{-a_2 t} U_{2n} + \dots + A_n e^{-a_n t} U_{nn}, \end{aligned}$$

$A_1, A_2, \dots, A_n$ étant n constantes arbitraires d'intégration.

On voit déjà par là que la véritable solution est bien de la forme à laquelle nous avions été conduits dans les §§2 et 3. Mais la forme symétrique des équations (2) va nous en apprendre d'avantage. La quantité de chaleur envoyée par M_i à M_k doit être égale à celle que reçoit M_k de M_i, on a donc:

$$C_{ik} = + C_{ki}.$$

Envisageons donc la forme quadratique positive:

$$\Phi(U_1, U_2, \dots, U_n) = \Sigma C_{ik}(V_i - V_k)^2 + \Sigma C_i V_i^2$$

les équations (2) pourront s'écrire:

$$a U_i = \frac{1}{2} \frac{d\Phi}{dU_i}. \tag{2 bis}$$

Si nous regardons $U_1, U_2, \dots, U_n$ comme les coordonnées d'un point dans l'espace à n dimensions, l'équation

$$\Phi = 1$$

représente un ellipsoïde puisque la forme Φ est positive.

Comment devrait-on procéder pour trouver les axes de cet ellipsoïde. Il faudrait précisément résoudre les équations (2 bis), éliminer les U_i entre ces équations, ce qui donnerait l'équation (3).

Il résulte de là que l'équation (3) a toutes ses racines réelles et positives. Ce résultat est connue de tous sous cette forme pour les ellipsoïdes dans l'espace ordinaire à 3 dimensions. Il est vrai encore dans le cas qui nous occupe, comme nous l'apprend la théorie des formes quadratiques.

En effet la forme Φ peut être décomposée en une somme de n carrés et nous écrirons cette décomposition sous la forme suivante:

$$\Phi = \alpha_1 \phi_1^2 + \alpha_2 \phi_2^2 + \dots + \alpha_n \phi_n^2$$

où:

$$\phi_p = U_{p1} U_1 + U_{p2} U_2 + \dots + U_{pn} U_n,$$

U_{p1}, U_{p2}, etc. étant des constantes.

Soit maintenant:

$$\Theta = U_1^2 + U_2^2 + \dots + U_n^2.$$

La théorie des formes quadratiques nous apprend que l'on peut choisir la décomposition de Φ de telle sorte que:

$$\Theta = \phi_1^2 + \phi_2^2 + \dots + \phi_n^2$$

ce qui entraîne les conditions:

$$U_{p1}^2 + U_{p2}^2 + \dots + U_{pn}^2 = 1, \tag{4}$$

$$U_{p1}U_{q1} + U_{p2}U_{q2} + \dots + U_{pn}U_{qn} = 0. \quad (p \gtrless q) \tag{5}$$

Si donc on écrit les n équations simultanées:

$$\phi_p = 1, \quad \phi_q = 0 \qquad (p \gtrless q) \tag{6}$$

ces équations admettront pour solution:

$$U_1 = U_{p1}, \quad U_2 = U_{p2}, \dots U_n = U_{pn}. \tag{7}$$

Or ces équations (6) entraînent les suivantes:

$$\frac{1}{2}\frac{d\Phi}{dU_i} = \alpha_p \phi_p \frac{d\phi_p}{dU_i} = \alpha_p \frac{d\phi_p}{dU_i}; \quad U_i = \frac{1}{2}\frac{d\Theta}{dU_i} = \phi_p \frac{d\phi_p}{dU_i} = \frac{d\phi_p}{dU_i}$$

d'où:

$$\frac{1}{2}\frac{d\Phi}{dU_i} = \alpha_p U_i.$$

Nous retrouvons les équations (2 bis). Les valeurs (7) des U et la valeur α_p de α nous représentent donc une solution de ces équations (2 bis).

Quelle est maintenant la signification de ces nombres $\alpha_1, \alpha_2, \dots, \alpha_n$. Supposons que ces nombres qui sont tous réels et positifs soient rangés par ordre de grandeur croissante.

Il est clair que α_1 sera le minimum du rapport:

$$\frac{\Phi}{\Theta} = \frac{\alpha_1\phi_1^2 + \alpha_2\phi_2^2 + \dots + \alpha_n\phi_n^2}{\phi_1^2 + \phi_2^2 + \dots + \phi_n^2}.$$

Si maintenant on suppose que les n variables U ne soient plus arbitraires, mais soient liées par la relation

$$\phi_1 = 0 \tag{8}$$

on verra que α_2 sera le minimum du rapport:

$$\frac{\Phi}{\Theta}=\frac{\alpha_2\phi_2^2+\dots+\alpha_n\phi_n^2}{\phi_2^2+\dots+\phi_n^2}.$$

Si les U sont liées par les deux relations

$$\phi_1=\phi_2=0, \tag{8 bis}$$

α_3 sera le minimum du rapport:

$$\frac{\Phi}{\Theta}=\frac{\alpha_3\phi_3^2+\dots+\alpha_n\phi_n^2}{\phi_3^2+\dots+\phi_n^2}$$

et ainsi de suite.

L'analogie avec l'analyse du §2 est évidente; il suffit pour retrouver cette analyse de passer à la limite comme l'a fait Fourier.

Les équations (1) sont analogues aux équations de Fourier:

$$\frac{dV}{dt}=a^2\Delta V,\quad \frac{dV}{dn}+hV=0.$$

Les équations (2) et (2 bis) sont analogues aux équations qui définissent les fonctions U et qui s'écrivent:

$$\Delta U+kU=0,\quad \frac{dU}{dn}+hU=0.$$

Les nombres α sont analogues aux nombres k.

La forme Φ est analogue à l'intégrale que nous avons appelée B dans le §2 et la forme Θ à l'intégrale que nous avons appelée A.

L'équation (4) est analogue à l'équation:

$$\int U_n^2 d\tau=1$$

et l'équation (5) (qui exprime que les axes de notre ellipsoïde sont rectangulaires) à l'équation:

$$\int U_n U_p d\tau=0. \qquad (n\gtrless p)$$

Il est inutile de pousser plus loin cette comparaison, on comprend suffisamment la parfaite identité des raisonnements, bien que ceux-ci soient parfaitement rigoureux dans le cas du présent paragraphe, où l'infini n'intervient pas, et qu'ils soient au contraire sujets à de graves objections dans le cas du §2.

Ce n'est pas seulement dans l'étude du problème de Fourier qu'on est conduit à ces considérations; on obtiendrait des résultats tout à fait analogues en envisageant au même point de vue les autres problèmes de Physique Mathématique.

Dans tous ces problèmes on a à intégrer des équations linéaires aux dérivées partielles. Ces équations ont partout la même origine. Les lois du phénomène véritable sont exprimées par des équations linéaires aux différentielles ordinaires, où la seule variable indépendante est le temps et où les inconnues sont en très grand-nombre; chacune de ces inconnues en effet, représente la valeur d'une certaine quantité relative à l'une des molécules du corps. Le nombre de ces inconnues est donc kn, n étant le nombre des molécules du corps, et k le nombre des quantités relatives à chaque molécule. C'est par un véritable passage de la limite qu'on passe ensuite de l'hypothèse moléculaire à celle de la matière continue et des équations différentielles ordinaires aux équations aux dérivées partielles.

Si donc on revient momentanément à l'hypothèse moléculaire, on n'a plus affaire qu'à des équations linéaires ordinaires à coefficients constants, et la seule difficulté provient du très grand nombre de ces équations. Mais il y a plus; ces équations présenteront presque toujours la symétrie que nous avons observée dans les équations (1) et on sera encore conduit à envisager une forme quadratique et tout sera ramené à la décomposition de cette forme en carrés.

Je n'en donnerai qu'un exemple; j'envisagerai les équations de l'élasticité. Soient x, y, z les coordonnées d'une molécule quelconque, dans l'état d'équilibre lorsque les forces extérieures appliquées au corps sont nulles; soient $x+u$, $y+v$, $z+w$ les coordonnées de cette même molécule lorsque le corps élastique est déformé sous l'action de forces extérieures; soit Φ la fonction des forces relative aux forces élastiques; soient X, Y, Z les trois composantes de la force extérieure appliquée à la molécule considérée. Les équations d'équilibre s'écriront alors:

$$\frac{d\Phi}{du} = X, \quad \frac{d\Phi}{dv} = Y, \quad \frac{d\Phi}{dw} = Z. \tag{9}$$

Comme u, v, w sont très-petits, nous pouvons développer Φ suivant les puissances de ces quantités et négliger les puissances d'ordre supérieure à 2. Les termes du 1^er^ degré doivent être nuls, puisque l'équilibre normal est atteint pour:

$$u = v = w = 0.$$

Je puis supposer que le terme tout connu est également nul; puisque Φ n'est déterminé qu'à une constante près.

En résumé Φ sera une forme quadratique par rapport aux u, v, w, et cette forme sera positive parce que l'équilibre normal doit être stable.

Les équations (9) seront donc linéaires en u, v, w. Le nombre de ces équations est le même que celui des inconnues u, v, w; il est égal à n, si le nombre des molécules est $\frac{n}{3}$. Afin de reprendre les mêmes notations que tout à l'heure, nous appellerons les n variables, U_1, U_2, ..., U_n; alors si les trois coordonnées d'une molécule que nous appelions tout à l'heure u, v, w, s'appellent maintenant U_p, U_{p+1}, U_{p+2}, nous appellerons de même X_p, X_{p+1} et X_{p+2} les composantes de la force extérieure appliquée à cette molécule, composantes que nous appelions tout à l'heure X, Y, Z et les équations (9) deviendront:

$$\frac{d\Phi}{dU_p} = X_p. \qquad (p = 1, 2, \ldots, n) \tag{9 bis}$$

Nous décomposerons la forme Φ en carrés comme nous l'avons fait tout à l'heure et nous retrouverons les formules:

$$\begin{aligned}
\Phi &= \alpha_1\phi_1^2 + \alpha_2\phi_2^2 + \ldots + \alpha_n\phi_n^2,\\
\phi_p &= U_{p1}U_1 + U_{p2}U_2 + \ldots + U_{pn}U_n,\\
\Theta &= \phi_1^2 + \phi_2^2 + \ldots + \phi_n^2,\\
\Theta &= U_1^2 + U_2^2 + \ldots + U_n^2.
\end{aligned}$$

Les équations (9 bis) deviennent alors:

$$\sum_{k=1}^{k=n} \alpha_k\phi_k U_{kp} = X_p \qquad (p = 1, 2, \ldots, n) \tag{9 ter}$$

Multiplions la première de ces équations par U_{i1}, la seconde par U_{i2}, ..., la n^e par U_{in} et ajoutons. En tenant compte des équations (4) et (5) il viendra:

$$\alpha_i\phi_i = U_{i1}X_1 + U_{i2}X_2 + \ldots + U_{in}X_n, \qquad (i = 1, 2, \ldots, n) \tag{10}$$

Multiplions maintenant la première des équations (10) par $\frac{U_{1\kappa}}{\alpha_1}$, la seconde par $\frac{U_{2\kappa}}{\alpha_2}$ ...., la n^e par $\frac{U_{n\kappa}}{\alpha_n}$; en vertu des équations d'orthogonalité (4) et (5) ou plutôt des équations

$$\begin{aligned}
U_{1p}^2 + U_{2p}^2 + \ldots + U_{np}^2 &= 1,\\
U_{1p}U_{1q} + U_{2p}U_{2q} + \ldots + U_{np}U_{nq} &= 0,
\end{aligned}$$

qui comme on le sait leur sont équivalentes; il viendra:

$$U_\kappa = \frac{U_{1\kappa}}{\alpha_1}\theta_1 + \frac{U_{2\kappa}}{\alpha_2}\theta_2 + \ldots + \frac{U_{n\kappa}}{\alpha_n}\theta_n$$

où l'on a posé pour abréger:

$$\theta_i = U_{i1}X_1 + U_{i2}X_2 + \ldots + U_{in}X_n.$$

Les équations (9) sont donc ainsi résolues.

Il est clair qu'une pareille solution ne peut être que théorique, comme l'était déjà la solution du problème de Fourier par l'intégration des équations (1). Le nombre immense des équations (1) comme celui des équations (9) s'opposerait absolument aux calculs. Mais cette solution purement théorique peut mettre sur la voie de la solution véritable.

Passons à la limite et abandonnons l'hypothèse moléculaire pour celle de la matière continue. Nos équations (1) ou (9) deviendront des équations aux dérivées partielles; nos formes quadratiques Φ et Θ deviendront des intégrales analogues à celles que nous avons appelées A et B dans le §2.

Notre procédé pourra s'appliquer sans autre changement; au lieu de décomposer les formes Φ et Θ en carrés, nous aurons à chercher les minima successifs de leur rapport ou plutôt ceux du rapport des intégrales A et B qui les remplacent. On passera ainsi d'une analyse analogue à celle de ce paragraphe à une analyse tout à fait semblable à celle du §2.

§6.—*Existence des Fonctions U_n.*

L'existence des fonctions U_n peut être maintenant regardée comme démontrée au moins au point de vue physique. La fonction U_1 sera une fonction qui devra prendre les valeurs

$$U_{11},\ U_{12},\ \ldots,\ U_{1n}$$

aux différents points occupés par les molécules

$$M_1,\ M_2,\ \ldots,\ M_n.$$

La fonction U_2 devra prendre en ces mêmes points les valeurs

$$U_{21},\ U_{22},\ \ldots,\ U_{2n}$$

et ainsi de suite.

Nous savons de plus que les valeurs de la fonction U_1 par exemple devront satisfaire aux équations (2 bis) du § précédent que j'écrirai

$$\frac{1}{2}\frac{d\Phi}{dU_i} = \alpha_1 U_i. \tag{1}$$

La fonction U_1 n'est définie ainsi, il est vrai que pour n points de l'espace, à savoir les n points occupés par nos n molécules; mais comme ces molécules sont très nombreuses et très rapprochées les unes des autres, on pourra calculer par interpolation la fonction U_1 pour tous les autres points intérieurs au corps.

On pourra à la vérité trouver ainsi deux valeurs différentes pour la fonction U_1 si l'on adopte deux règles d'interpolation différentes; mais les différences seront du même ordre de grandeur que la distance qui sépare deux molécules et par conséquent négligeables au point de vue physique.

La fonction U_1 ainsi définie satisfera approximativement aux équations de Fourier:

$$\Delta U_1 + k_1 U_1 = 0, \quad \frac{dU_1}{dn} + hU_1 = 0 \tag{2}$$

que l'on obtient en partant des équations (2 bis) et en passant à la limite. L'erreur commise en remplaçant les équatious (1) par les équations (2) sera du même ordre de grandeur que la distance qui sépare deux molécules.

Il y a à cela toutefois une condition, c'est que les dérivées de la fonction U_1 soient finies; on n'aurait plus le droit de passer des équations (1) aux équations (2) étaient du même ordre de grandeur que l'inverse de la distance qui sépare deux molécules.

Il nous resterait donc à établir que ces dérivées sont bien finies; c'est-à-dire:

1°. Que la différence $U_{1i} - U_{1k}$ est du même ordre de grandeur que la distance des molécules M_i et M_κ.

2°. Plus généralement, soit M_α une molécule quelconque de coordonnées x, y, z, soient $M_\beta, M_\gamma, \ldots, M_\lambda$ un certain nombre de molécules très voisines de M_α et dont les coordonnées soient respectivement,

$$x+\xi_\beta,\ y+\eta_\beta,\ z+\zeta_\beta;\ x+\xi_\gamma,\ y+\eta_\gamma,\ z+\zeta_\gamma;\ \ldots;\ x+\xi_\lambda,\ y+\eta_\lambda,\ z+\zeta_\lambda.$$

Soit $P(\xi, \eta, \zeta)$ un polynôme quelconque de degré inférieur à m en ξ, η, ζ. Soient enfin $\alpha, \beta, \gamma, \ldots, \lambda$ un certain nombre de coefficients relatifs aux diverses molécules $M_\alpha, M_\beta, \ldots, M_\lambda$; et supposons que ces coefficients satisfassent à la condition suivante:

$$\alpha P(\xi_\alpha, \eta_\alpha, \zeta_\alpha) + \beta P(\xi_\beta, \eta_\beta, \zeta_\beta) + \gamma P(\xi_\gamma, \eta_\gamma, \zeta_\gamma) + \ldots + \lambda P(\xi_\lambda, \eta_\lambda, \zeta_\lambda) = 0$$

et cela quel que soit le polynôme P pourvu que son degré soit inférieur à m (j'écris par symétrie $P(\xi_\alpha, \eta_\alpha, \zeta_\alpha)$ au lieu de $P(0,0,0)$).

Si ces conditions sont remplies, nous aurions à établir que:

$$\alpha U_{1\alpha} + \beta U_{1\beta} + \ldots . + \lambda U_{1\lambda}$$

est du même ordre de grandeur que les puissances m^{es} des quantités ξ, η, ζ.

Cela ne serait sans doute pas impossible; je n'aurais en effet pour établir ces divers points qu'à traduire dans le langage de l'hypothèse moléculaire l'analyse du §4.

C'est ce que je me réserve de faire dans un mémoire ultérieur qui pourra être regardé comme la suite de celui-ci.

Je pourrai dire alors que les conclusions des §§2, 3 et 4 sont démontrées d'une façon rigoureuse au point de vue physique. Peut être même est-il permis d'espérer que, par une sorte de passage à la limite, on pourra fonder sur ces principes une démonstration rigoureuse même au point de vue analytique.

PARIS, le 19 Mars, 1889.

Singular Solutions of Ordinary Differential Equations.

By Henry B. Fine.

The method best suited to a general investigation of the conditions of occurrence of singular solutions of differential equations and of the properties of these solutions would seem to be that introduced by Briot and Bouquet in their study of ordinary solutions, and developed very beautifully in their memoir, *Propriétés des Fonctions définies par des Equations Différentielles.**

By this method the theory of singular solutions may be based immediately on the differential equation, and any use, direct or indirect, of the notion of the complete primitive be avoided; it is not required to restrict the variables or coefficients of the equation to real values, and account may be taken of the region, not only of points which are ordinary points for the equation, but of such as are its singular points † as well.

There are obvious objections to the use of the notion of a complete primitive in any investigation of singular solutions which seeks to be general or rigorous. In the first place it is an inversion of the natural order of investigation, since what is sought is the conditions which a differential equation $f(x, y, p) = 0$ (or one of higher order) must fulfil in order that a curve not properly contained in the system which it defines may satisfy it; not the characteristics of a system of curves defined by some equation $\phi(x, y, \alpha) = 0$ between x, y, and a parameter α.

But in the second place, while the curve system $\phi(x, y, \alpha) = 0$ generally has an envelope, and therefore the differential equation $F(x, y, p) = 0$ by which

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* Journal de l'Ecole Polytechnique, Cah. 36.

† Poincaré has given this name to points at which $\frac{dy}{dx}$ is indeterminate; in the case of the equation $\frac{dy}{dx} = \frac{Y}{X}$, for instance, to the points of intersection of $X = 0$, $Y = 0$.

it may also be defined, a singular solution—the *general* differential equation $f(x, y, p) = 0$ has no singular solution—the curve system which it defines, no envelope. If the equation be of the n^{th} degree, there will pass through any ordinary point x_0, y_0 n curves which satisfy it; but *all* that it gives to define each of these curves is a development of y in positive powers of x. The coefficients in any of these developments are, it is true, rational functions of x_0, y_0 and of one or other of the roots, p_0, of $f(x_0, y_0, p) = 0$, and therefore algebraic functions of a single parameter. But an envelope is not to be had by variation of this parameter; for, at the points of $\text{Disct}_p f(x, y, p) = 0$, the curve which, if any, is the envelope of the system defined by $f(x, y, p) = 0$, the development generally becomes divergent and so ceases altogether to represent a solution of the equation.*

Briot and Bouquet have shown that all ordinary solutions of the equation $f(p, y, x) = 0$ which pass through a given point x_0, y_0, and have there $p = p_0$ in common, may by suitable transformations be reduced to the form $y = Vvx^m$; where V is a determinate function of v and x, and v is connected with x by a differential equation of the form $\frac{dv}{dx}x = \phi(v, x)$. In the first of the following sections it is proven that if $f(p, y, x) = 0$ have a singular solution and x_0, y_0 lie on it, among the functions V determined by the Briot-Bouquet method will be one which corresponds to the singular solution—but that the differential equation between v and x which corresponds to this value of V degenerates into the form

$$\frac{dv}{dx}x = \frac{\psi(v, x)}{\psi(v, x)}\chi(v, x),$$

generally satisfied by $\psi(v, x) = 0$ only. This result brings out clearly the exact relation in which the singular solution stands to the differential equation itself. In the strict sense it is not contained in the equation any more than any arbitrary function $\phi(x, y) = 0$ is contained in the equation $\frac{dy}{dx} = \frac{\phi(x, y)}{\phi(x, y)}\psi(x, y)$. Yet

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*Darboux was apparently the first to call attention to and explain the paradox involved in the old (Lagrange) theory of singular solutions, according to which, since every differential equation of the first order is to be regarded as having a complete primitive of the form $\phi(x, y, a) = 0$, and $\phi(x, y, a) = 0$ as generally having an envelope, the differential equation should generally have a singular solution. *Vid.* Comptes rendus, t. 70 and 71, and Mémoires de l'Institut, t. XXVII.

it takes the place of one of the two solutions ordinarily occurring for values x_0, y_0, p_0 of x, y, p, which satisfy the two equations $f = 0$, $\frac{\partial f}{\partial p} = 0$.

In the second and principal section of the paper, the theory developed in the first is extended to equations of higher orders.

Section 3 contains a discussion of the singular solutions of those differential equations of the n^{th} order whose general solution is obtainable in the form $\phi(x, y, \alpha, \beta, \gamma, \ldots \nu) = 0$, with special reference to the geometrical interpretation of the equation which, algebraically speaking, contains the singular solution. This section is added in deference to a usage which has become almost classical, of regarding singular solutions from the double standpoint of the differential equation itself and its complete primitive. The objection to basing the general theory of singular solutions on that of systems of curves defined by equations of the form $\phi(x, y, \alpha, \beta, \gamma, \ldots \nu) = 0$ has already been stated; the curve system and the differential equation which represents it may nevertheless be of sufficient interest on their own account to merit the space given them. ϕ, it should be added, is supposed to be a one-valued function of x, y within the region in which it is studied, and a rational function of n parameters $\alpha, \beta, \ldots \nu$, or, more generally, of m parameters connected by $m - n$ algebraic equations.

§1.

Let $f(x, y, p)$ be any rational integral function of x, y and p, irreducible with respect to p and containing no mere x, y factor: though the discussion which is to follow applies as well to any function of the form $f_0 p^m + f_1 p^{m-1} + \ldots f_m$, if $f_0, f_1, \ldots$ be holomorphic functions of x and y with a common region of convergence and x, y throughout the discussion be restricted to that region.

The differential equation

$$f(x, y, p) = 0 \qquad (1)$$

defines a system of curves, m of which, speaking generally, pass through any point x_0, y_0, and for each of these m curves a development $y - y_0 = \phi(x - x_0)$ is to be had by the following method, first given by Briot and Bouquet.

It is for the sake of convenience and definiteness of statement only that we

call x_0, y_0 a point, for complex as well as real values of x and y are taken account of in the discussion.

Let it be assumed that the coefficient of the highest power of p in f does not vanish for $x = x_0$, $y = y_0$: the values of p corresponding to $x_0 y_0$ and given by the algebraic equation $f(x_0, y_0, p) = 0$, are then all finite; call any one of them p_0.

(A method for dealing with the case when the coefficient of the highest power of p vanishes (which Briot and Bouquet do not consider) is given in my paper "On Functions defined by Differential Equations, etc.,"* and is also stated in §2 of the present paper.)

First of all make in $f = 0$ the substitutions

$$\begin{aligned} x &= x_0 + x' \\ y &= y_0 + p_0 x' + y' \\ p &= p_0 + p', \end{aligned} \tag{2}$$

and develop by Taylor's theorem. We then have

$$0 = F(x', y', p') \equiv \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}\right)_0 x' + \left(\frac{\partial f}{\partial y}\right)_0 y' + \left(\frac{\partial f}{\partial p}\right)_0 p' + \ldots \tag{3}$$

where p' as well as y' vanishes with x'.

1. Suppose $\left(\frac{\partial f}{\partial p}\right)_0$ and $\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}\right)_0$ to be both different from zero.

Equation (3) then gives for p' a single development in integral powers of x', y', call it $\phi(x', y')$; and the equation $\frac{dy'}{dx'} = \phi(x', y')$ defines y' as a holomorphic function of x',† which is the only solution of (3) vanishing with x' in this the case of usual occurrence.

2. Let $\left(\frac{\partial f}{\partial p}\right)_0 = 0$, but $\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}\right)_0 \neq 0$.

In this case p_0 is a repeated root of the equation $f(x_0, y_0, p) = 0$. To have the general case of repeated roots before us, let the multiplicity of the root p_0 be supposed to be r; then $\left(\frac{\partial f}{\partial p}\right)_0 = \left(\frac{\partial^2 f}{\partial p^2}\right)_0 = \ldots = \left(\frac{\partial^{r-1} f}{\partial p^{r-1}}\right)_0 = 0$, but $\left(\frac{\partial^r f}{\partial p^r}\right)_0 \neq 0$.

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* Amer. Journal XI, 4.

† By Cauchy's theorem. *Vid.* Briot and Bouquet in Journ. de l'Ecole Pol. Cah. 36, and in Théorie des Fonc. Ellipt. p. 325.

The terms of lowest degree in (3) are then $\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)_0 x'$ and $\frac{1}{r!}\left(\frac{\partial^r f}{\partial p^r}\right)_0 p'^r$.

Let $x' = x''^r$. Then (3) gives r developments for p' in integral powers of x'' and y', the first term in each development being one of the values of

$$\left(-\frac{\frac{\partial f_0}{\partial x_0} + \frac{\partial f_0}{\partial y_0} p_0}{\frac{1}{r!}\frac{\partial^r f_0}{\partial p_0^r}}\right)^{\frac{1}{r}} x''.$$

Each of these developments on being substituted for p' in the equation $\frac{dy'}{dx''} = rx''^{r-1} p'$, gives y' as a holomorphic function of x'',* developable, therefore, in integral powers of $x'^{\frac{1}{r}}$.

There are then in this case r solutions connected cyclicly, the lowest power of x' in each development for y' being $x'^{\frac{r+1}{r}}$. Of the n distinct curves defined by $f(x, y, p) = 0$ for the region of an ordinary point x_0, y_0, r have been replaced by the cyclicly connected branches of a single curve on which x_0, y_0 is a singular point of the cusp species.

It follows immediately that

$\mathrm{Disct}_p\, f(x, y, p) = 0$ is in the general case a locus of cusps on the curves defined by the equation $f(x, y, p) = 0$.

The possession of cusps is thus shown to be *characteristic* of the curves defined by the general differential equation of the first order when its degree is higher than 1.

It is obvious that when the coefficients of f are real, the curve $\mathrm{Disct}_p\, f = 0$ may be used to separate between the regions of real and imaginary solutions of $f = 0$; for as the point x_0, y_0 crosses $\mathrm{Disct}_p f = 0$, two of the corresponding values of p_0 commonly pass from real to conjugate imaginary values, or vice versa. Thus for the equation

$$p^2 + x^2 + y^2 - 1 = 0$$

the circle $x^2 + y^2 - 1 = 0$ divides the x, y plane into two regions, in one of which the solutions are always real, in the other always imaginary.

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* By Cauchy's theorem. *Vid.* foot-note on p. 298.

3. Let $\left(\frac{\partial f}{\partial p}\right)_0 = 0$ and $\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} p\right)_0 = 0$.

There is generally a finite number only of such points, called by Poincaré the "singular points"* of the equation $f = 0$.

Again to have the general case before us, suppose any additional coefficients of F to vanish.

Now since x', y', p' vanish together, any solution of equation (3) may evidently be thrown into the form

$$y' = p'x'v, \tag{4}$$

where v is a function of x' which takes a finite value v_0, different from zero when $x' = 0$.

On substituting this value of y' in (3), that equation goes over into an equation in p', x' with coefficients which are functions of v.

To get the terms of lowest degree in this p', x' equation follow the Puiseux method or any other method applicable to algebraic equations in two variables.

There may be a number of groups of terms of the same degree in the equation, each of which gives for μ (the degree of p' in respect to x') a value such that the terms of the group are the lowest in the equation.

Consider any one such group, for which $\mu = \frac{r}{s}$.

Make the substitutions

$$x' = x''^s,\ p' = Vx''^r,\ v = v_0 + v'. \tag{5}$$

V has a finite value different from zero when $x'' = 0$; and v_0, the initial value of v, is equal to $\frac{s}{r+s}$, as may readily be seen by comparing the equations $p' = Vx'^{\frac{r}{s}}$ and $y' = p'x'v$ for $x' = 0$.

The transformed equation—call it $\phi(x'', v', V) = 0$—then gives for V one or more developments in integral powers of v' and x''. It is assumed that the equation $\phi(0, 0, V) = 0$ has no multiple roots.

Let $V = a + bx'' + cv' + \ldots.$ be one of these series.

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* *Vid.* Poincaré in Journal de Math. pure et appliquées, III, 7, 8; IV, 1. These singular points will not be confounded with the singular points of Fuchs and other writers on linear differential equations, the points of discontinuity of coefficients of the equation.

The equation connecting v' and x'' is then obtained by making the substitutions (5) in the equation $y' = p'x'v$. It is

$$x''\frac{dv'}{dx''} = -\frac{(r+s)\,Vv' + \left(\frac{s}{r+s} + v'\right)x''\frac{\partial V}{\partial x''}}{V + \left(\frac{s}{r+s} + v'\right)\frac{\partial V}{\partial v'}}\,; \tag{6}$$

or, replacing V by the series obtained for it,

$$x''\frac{dv'}{dx''} = -\frac{(r+s)^2 a}{(r+s)\,a + sc}\,v' - \frac{sb}{(r+s)\,a+sc}\,x'' + \ldots.$$

Now this is the form to which the solutions of the equation of the first degree, $\frac{dy}{dx} = \frac{Y}{X}$, can be reduced at points where $X = Y = 0$, the "singular points" of this equation. And by the theory of these solutions,* when $-\frac{(r+s)^2 a}{(r+s)\,a+sc}$ is not a positive integer, equation (6) admits always of a monodrome integral vanishing with x''—there being besides an infinite number of non-monodrome integrals when the real part of this coefficient is positive, no other when it is negative.

When, on the other hand, $-\frac{(r+s)^2 a}{(r+s)\,a+sc}$ is a positive integer, (6) can be transformed into an equation of the form $t\frac{du}{dt} = u + bt + \ldots.$ which has an infinite number of monodrome integrals when $b = 0$, and no monodrome but an infinite number of non-monodrome integrals when $b \neq 0$.

Finally, if $(r+s)\,a + sc = 0$, (6) is of the form

$$x''\frac{dv'}{dx''} = \frac{a_2 v' + b_2 x'' + \ldots.}{a_1 v' + b_1 x'' + \ldots.}$$

which in general has no monodrome integral vanishing with x''. This case, it should be added, presents itself when $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}p$ vanishes for two or any finite number of consecutive points of $\mathrm{Disct}_p f = 0$; that is to say, when singular points

* *Vid.* Briot and Bouquet in earlier sections of the memoir already referred to, and Poincaré, Journal de Math., III, 7.

of the equation $f = 0$ fall together. The corresponding curve-elements of $\mathrm{Disct}_p f = 0$ satisfy the equation $f = 0$, but the elements which lie to either side of this do not satisfy this equation (*vid.* 4 inf.)

To every solution of (6) there of course corresponds one solution of (3) which may be obtained by substituting for v' its value in the equation $y' = Vvx'^{\frac{r+s}{s}}$.

If, on the other hand, V_0 is a multiple root of $\phi(0, 0, V) = 0$, the corresponding series for V obtained from $\phi(x'', v', V) = 0$ may involve fractional powers of x'' and v', and, to render these powers integral, substitutions of the form $x'' = x'''^p$, $v' = v''^q$ are necessary. Equation (6) is replaced by a similar equation between v'' and x''', viz.

$$x''' \frac{dv''}{dx'''} = -\frac{p(r+s)Vv''^q + \left(\frac{s}{r+s} + v''^q\right) x''' \frac{\partial V}{\partial x'''}}{qVv''^{q-1} + \left(\frac{s}{r+s} + v''^q\right) \frac{\partial V}{\partial v''}}. \tag{6'}$$

Singular Solutions.

4. Let $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} p = 0$ at every point of $\mathrm{Disct}_p f = 0$, or if this curve be reducible, at every point of at least one of its branches through x_0, y_0.

This case is altogether exceptional; it requires the coefficients of f to be so taken that the eliminant of $f = 0$, $\frac{\partial f}{\partial p} = 0$, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} p = 0$ with respect to any two of the variables x, y, p shall vanish identically. But if it occur,

$\mathrm{Disct}_p f = 0$ satisfies the equation $f = 0$.

First suppose that the surface which the equation $f(x, y, p) = 0$ defines, when x, y, p are regarded as the coordinates of a point in space, has no singularity at the point x_0, y_0, p_0.

Since $\left(\frac{\partial f}{\partial p}\right)_0 = 0$, the equation of the tangent plane to the surface at this point is

$$\left(\frac{\partial f}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) = 0.$$

This plane cuts the xy plane in the tangent line to the projection of the

intersection of the surfaces $f = 0$ and $\dfrac{\partial f}{\partial p} = 0$. The slope of this tangent line is therefore $-\dfrac{\left(\dfrac{\partial f}{\partial x}\right)_0}{\left(\dfrac{\partial f}{\partial y}\right)_0}$, that is to say, p_0, since $\left(\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} p\right)_0 = 0$.

Thus at each of its points $\text{Disct}_p f = 0$—which is the projection of the intersection of $f = 0$, $\dfrac{\partial f}{\partial p} = 0$—touches a curve of the f-system, and therefore satisfies the equation $f = 0$.

Second, suppose x_0, y_0, p_0 to be a singular point of the surface $f(x, y, p) = 0$. The substitutions (2) which transformed f into F also transform $\dfrac{\partial f}{\partial p}$ into $\dfrac{\partial F}{\partial p'}$ and $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} p$ into $\dfrac{\partial F}{\partial x'} + \dfrac{\partial F}{\partial y'} p'$.

Again, by virtue of the substitutions (4) and (5) we have the identity

$$F(x', y', p') \equiv X \phi(x'', v', V),$$

where X is an irrelevant factor of the form $x''^p v^q V^t$.

Differentiating with respect to v, V and x'' we obtain

$$\frac{\partial F}{\partial y'} V x''^{r+s} = X \frac{\partial \phi}{\partial v} + \frac{dX}{\partial v} \phi,$$

$$\frac{\partial F}{\partial y'} v x''^{r+s} + \frac{\partial F}{\partial p'} x''^r = X \frac{\partial \phi}{\partial V} + \frac{\partial X}{\partial V} \phi,$$

$$(r+s) \frac{\partial F}{\partial y'} V v x''^{r+s-1} + r \frac{\partial F}{\partial p'} V x''^{r-1} + s \frac{\partial F}{\partial x''} x''^{s-1} = X \frac{\partial \phi}{\partial x''} + \frac{\partial X}{\partial x''} \phi;$$

whence

$$\left.\begin{aligned} \frac{\partial F}{\partial p'} &= \frac{X}{V x''^r} \left(\frac{\partial \phi}{\partial V} V - \frac{\partial \phi}{\partial v} v \right) + X_1 \\ \frac{\partial F}{\partial x'} + \frac{\partial F}{\partial y'} p' &= \frac{X}{x''^s} \left(\frac{x''}{s} \frac{\partial \phi}{\partial x''} + \frac{\partial \phi}{\partial v}(1 - v) - \frac{r}{s} \frac{\partial \phi}{\partial V} V \right) + X_2 \end{aligned}\right\} \qquad (7)$$

where X_1 and X_2 are terms of the form $M\phi$ and therefore vanish with ϕ.

When, therefore, F, $\dfrac{\partial F}{\partial p'}$ and $\dfrac{\partial F}{\partial x'} + \dfrac{\partial F}{\partial y'} p'$ vanish together, and that not merely at the point $x' = y' = p' = 0$, but all along a common curve through

that point, then along this same curve

$$\begin{aligned} &\phi &&= 0, \\ &\frac{\partial\phi}{\partial V} V - \frac{\partial\phi}{\partial v} v &&= 0, \\ &\frac{x''}{s}\frac{\partial\phi}{\partial x''} + \frac{\partial\phi}{\partial v}(1-v) - \frac{r}{s}\frac{\partial\phi}{\partial V} V &&= 0. \end{aligned}$$

In this case, among the roots, $V = a + bx'' + cv' + \ldots$, of $\phi(x'', v', V) = 0$—each of which ordinarily leads, when substituted in (6), to a solution of the equation $F = 0$ (see 3)—is one which gives rise to the same development for v' in powers of x'', whether substituted in $\frac{\partial\phi}{\partial V} V - \frac{\partial\phi}{\partial v} v = 0$, or in

$$\frac{x''}{s}\frac{\partial\phi}{\partial x''} + \frac{\partial\phi}{\partial v}(1-v) - \frac{r}{s}\frac{\partial\phi}{\partial V} V = 0.$$

Call this development $v' = \psi(x'')$. It is, by the method of its derivation $\left(\text{from } \phi = 0 \text{ and } \frac{\partial\phi}{\partial V} V - \frac{\partial\phi}{\partial v} v = 0\right)$, the development which defines a branch of the curve $\text{Disct}_p f = 0$ passing through the point $x' = y' = p' = 0$.

But it also satisfies, in the algebraic sense, the equation (6).

For since $\phi = 0$,

$$\frac{\partial\phi}{\partial x''} dx'' + \frac{\partial\phi}{\partial v} dv + \frac{\partial\phi}{\partial V} dV = 0;$$

whence
$$\frac{\partial V}{\partial x''} = -\frac{\frac{\partial\phi}{\partial x''}}{\frac{\partial\phi}{\partial V}} \text{ and } \frac{\partial V}{\partial v'} = -\frac{\frac{\partial\phi}{\partial v'}}{\frac{\partial\phi}{\partial V}}.$$

So that equation (6) may be thrown into the form:

$$\begin{aligned} x''\frac{dv'}{dx''}\left(\frac{\partial\phi}{\partial V} V - \frac{\partial\phi}{\partial v} v\right) &= -(r+s)\frac{\partial\phi}{\partial V} Vv' + \frac{\partial\phi}{\partial x''} vx'' \\ &= s(1-v)\left(\frac{\partial\phi}{\partial V} V - \frac{\partial\phi}{\partial v} v\right) + v\left(\frac{\partial\phi}{\partial x''} x'' + s\frac{\partial\phi}{\partial v}(1-v) - r\frac{\partial\phi}{\partial V} V\right). \quad (8) \end{aligned}$$

For the root V of $\phi = 0$ under consideration, therefore—the root which gives rise to the development $v' = \psi(x'')$—the equation (6) is satisfied; not, how-

ever, as a differential equation, but through the vanishing of factors algebraic in x'', v', V which appear in all its terms.

Equation (8) brings out very clearly the relation in which the discriminant curve stands to the differential equation, when it satisfies this equation. It is no more a part of the solution of the equation, strictly speaking, than the arbitrary function χ is of the equation $\frac{dy}{dx}\chi = \Phi\chi$. Yet it takes the place of one of the f-curves through x_0, y_0 to which the method of 3 generally leads. For to it there corresponds a root (V) of $\phi = 0$, though all the roots of this equation usually yield proper solutions of $f = 0$, and the equation (6) to which this root leads has no other solution vanishing with x'' than $v' = \psi(x'')$.

The discriminant curve may at particular points have contact of an order higher than the first with curves of the system $f = 0$. The corresponding V_0 is then a multiple root of $\phi(0, 0, V) = 0$, and one is led to equations (6′) between v'' and x''' instead of equations (6) between v' and x''. A mere repetition of the argument of the present section shows, however, that the v'' belonging to the discriminant curve satisfies its equation again in the algebraic sense only.

It is of course possible for the discriminant curve to be at the same time a curve of the system $f = 0$. V_0 is then again a multiple root of $\phi(0, 0, V) = 0$; and there will be two developments for V which, with their corresponding equations (6′) and $y' = Vvx'^{\frac{r+s}{s}}$, define the same curve—once as the discriminant curve and again as a curve of the system $f = 0$.

To suppose of a differential equation that it is satisfied by the discriminant curve, or a branch of this curve, is the same thing, algebraically speaking, as to suppose that it has an infinite number of singular points. As was noticed at the end of 3, when two or more consecutive points of $\text{Disct}_p f = 0$ are singular points of $f = 0$, the corresponding element of $\text{Disct}_p f = 0$ satisfies $f = 0$, but that element only; if then all points of a branch of $\text{Disct}_p f = 0$ are singular points of $f = 0$, this branch must itself satisfy the equation.

§2.

There is no essential difficulty in extending the preceding theory to equations of higher orders.

Consider the equation

$$f \equiv f_0 y_n^m + f_1 y_n^{m-1} + \ldots + f_m = 0, \qquad (1)$$

where $f_0, f_1, \ldots f_m$ are holomorphic functions of $x, y, y_1, \ldots y_{n-1}$ within common regions of convergence.

This equation defines an n-ply infinite system of curves of which there are, generally speaking, m for each set of values $x_0, y_0, y_1^0, \ldots y_{n-1}^0$ of $x, y, y_1, \ldots y_{n-1}$, within the common regions of convergence of $f_0, f_1, \ldots f_n$; i. e. m which pass through the point x_0, y_0 and there have with one another contact of the $(n-1)^{\text{th}}$ order of which the elements are $y_1^0, y_2^0, \ldots y_{n-1}^0$.

The value of y_n belonging to each of these m curves is ordinarily to be obtained directly by solving (1) which is algebraic in y_n, and a development for the curve itself—of y in powers of x—is then to be had by first making in (1) the substitutions,

$$\left.\begin{aligned} x &= x_0 + x', \\ y &= y_0 + y_1^0 x' + y_2^0 \frac{x'^2}{2!} + \ldots y_n^0 \frac{x'^n}{n!} + y' \end{aligned}\right\} \tag{2}$$

and so transforming it into an equation in $x', y', y_1', \ldots y_n'$ in which y' and *all* the differential coefficients $y_1', y_2', \ldots y_n'$ vanish with x': viz. the equation

$$0 = F \equiv \left(\frac{\partial f}{\partial y_{n-1}} y_n + \frac{\partial f}{\partial y_{n-2}} y_{n-1} + \ldots \frac{\partial f}{\partial y} y_1 + \frac{\partial f}{\partial x}\right)_0 x' + \left(\frac{\partial f}{\partial y}\right)_0 y'$$
$$+ \left(\frac{\partial f}{\partial y_1}\right)_0 y_1' + \ldots + \left(\frac{\partial f}{\partial y_n}\right)_0 y_n' + \ldots, \tag{3}$$

and then from this equation deriving a development of y' in powers of x' by the method given in my paper already referred to—"On Functions defined by Differential Equations, etc." and also presented briefly later on in the present paper.

If, however, f_0, the coefficient of y_n^m in (1) vanishes, one or more of the values of y_n^0 are either infinite or cannot be determined by the method just used.

In this case transform (1) by the substitutions:

$$\left.\begin{aligned} x &= x_0 + x', \\ y &= y_0 + y_1^0 x' + y_2^0 \frac{x'^2}{2!} + \ldots + y_{n-1}^0 \frac{x'^{n-1}}{(n-1)!} + y' \end{aligned}\right\} \tag{4}$$

into an equation

$$\psi(x', y', \ldots y_{n-1}', y_n') = 0 .$$

in which y', $y'_1, \ldots . y'_{n-1}$ vanish with x'; and then, regarding $\psi = 0$ as an equation in x', y', $\ldots . y'_{n-1}$, y'_n, or in x', y', $\ldots . y'_{n-1}$, $\frac{1}{y'_n}$ according as f_n also does or does not vanish for $x = x_0$, $y = y_0$, $\ldots . y_{n-1} = y^0_{n-1}$, determine the degree of y' in respect to x' by the polygon construction given in the paper just mentioned.*

Inasmuch as not only f_0 but also other of the coefficients of f may vanish for $x = x_0$, $y = y_0$, $\ldots . y_{n-1} = y^0_{n-1}$, the degree of y' in respect to x'—call it μ—may admit of a number of determinations; that is to say, it may be possible in a number of ways to select groups of terms in $\psi = 0$ which a certain value of μ will render of the same degree with each other, and at the same time of lower degree than the remaining terms in this equation. To each of these values of μ will correspond one or more of the ψ-curves for which y', $\ldots . y'_{n-1}$ vanish with x'.

We consider the cases $\mu > n$, $\mu = n$, $\mu < n$, and in each case transform $\psi = 0$ into an equation in which, as in (3), the n^{th} differential coefficient as well as those of lower orders vanishes with the independent variable.

(a) If $\mu > n$, y'_n vanishes with x' and $\psi = 0$ already has the form $F = 0$. Indeed in this case since $y^0_n = 0$, the substitution (4) is identical with the substitution (2).

(b) If $\mu = n$, set $y' = \frac{vx'^n}{n!}$ in the group of lowest terms in ψ which corresponds to this value of μ, and after removing the common factor x'^n let $x' = 0$.

Each root v_0 of the resulting equation will be the y^0_n of one of the curves sought for, and for each $\psi = 0$ may be transformed into the form $F = 0$ by the substitution

$$y' = \frac{y^0_n}{n!} x'^n + y''. \tag{5}$$

The substitutions (4) and (5) together constitute a substitution (2).

(c) Finally if $\mu < n$, the substitution $x' = x''^m$—where m is the first integer greater than $\frac{n}{\mu}$— will transform $\psi = 0$ into an equation in which all the differential coefficients to the n^{th} inclusive vanish with x''.

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*I have been at pains to restate the treatment of the case $f_0 = 0$ with some fulness, since the statement of it in my former paper is careless and in one particular incorrect (*vid.* Am. Journ. XI, p. 321, line 18 from top).

Particular mention may be made in passing of the case when for $x = x_0$, $y = y_0, \ldots, y'_{n-1} = y^0_{n-1}$, the coefficient f_0 vanishes and that to the first degree in $x - x_0$, $y - y_0, \ldots$, while f_1 takes a value different from zero.

As the polygon construction readily shows, the one group of lowest terms in $\psi = 0$ then consists of the $\frac{1}{y'_n}$ and y'_{n-1} terms, and for this group $\mu = n - \frac{1}{2}$. The theorem* immediately follows:

The differential equation of the $(n-1)^{th}$ *order,* $f_0 = 0$, *defines a system of curves each of which has at every point along it a contact of order* $n-1$ *with a curve of the system* $f = 0$, *which there has a singularity of the cusp class characterized by the development:*

$$y = y_0 + y^0_1 x' + y^0_2 \frac{x'^2}{2!} + \ldots + y^0_{n-1} \frac{x'^{n-1}}{(n-1)!} + Ax'^{n-\frac{1}{2}} + Bx'^n + \ldots$$

In like manner if f_0 and f_1 both vanish to the first degree in $x - x_0$, $y - y_0, \ldots$ but f_2 does not vanish, one value of μ is $n - \frac{1}{3}$; and generally, if $f_0, f_1, \ldots f_{\kappa-1}$ all vanish, but f_κ does not, one value of μ is $n - \frac{1}{\kappa + 2}$.

(It may happen that there are no curves of the system $f = 0$ with the initial elements $x_0, y_0, \ldots y^0_{n-1}$. Thus no curve of the system

$$xy_2 = 1$$

passes through $x = y = 0$ and there has $y_1 = 0$.)

For the sake of definiteness and simplicity of statement, the discussion which follows will be confined to curves of the system for which $f = 0$ can be transformed into the form $F = 0$; that is, to those for which $\mu >$ or $= n$. Of course the curves for which $\mu < n$ admit of a precisely similar treatment; but in consequence of the substitution $x = x'^m$ the coefficients in their developments are differently related to the coefficients of f.

The reduction to the form $F = 0$ having been accomplished, the development for the curve—of y' in powers of x'—may be obtained as follows.

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* This theorem is also given by Goursat, this Journ. XI, 4.

Make in F the substitutions

$$\left.\begin{aligned} y'_{n-1} &= y'_n v_1 x', \\ y'_{n-2} &= y'_n v_1 v_2 x'^2, \\ &\cdots\cdots\cdots\cdots \\ y' &= y'_n v_1 v_2 \ldots . v_n x'^n \end{aligned}\right\} \tag{6}$$

where $v_1, v_2, \ldots . v_n$ are functions of x' which take finite values $v_1^0, v_2^0, \ldots . v_n^0$ different from zero when $x' = 0$.

In the resulting equation, freed from any factor which may be common to all its terms, determine the various groups of terms of lowest order in y'_n and x'. This is, of course, necessary only when certain of the coefficients in (3) vanish, in particular those of y'_n and x'; in which case more than one of the curves or curve-branches of the f-system may have the initial elements $x_0, y_0, \ldots . y_n^0$, and (3) must give rise to a development for each of them.

Suppose that for any particular group of terms of lowest order the degree of y'_n in respect to x' is $\frac{r}{s}$; to obtain the corresponding developments make then the further substitutions

$$x' = x''^s,\ y'_n = Vx''^r,\ v_\kappa = v_\kappa^0 + v'_\kappa \tag{7}$$

—where again V takes a finite value V_0 different from zero when $x'' = 0$—and thus transform (3) into an equation in $x'', v'_1, v'_2, \ldots . , v'_n, V$; viz.

$$\phi\,(x'', v'_1, v'_2, \ldots . v'_n, V) = 0. \tag{8}$$

The initial value v_κ^0 of v_κ may readily be shown to be $\frac{s}{r + \kappa s}$.

From the equation $\phi = 0$ developments are to be obtained for V in integral powers of x'' and $v'_1, v'_2, \ldots . v'_n$—one for each of the curves the degree of whose y'_n in respect to x' is $\frac{r}{s}$. It is assumed that $\phi\,(0, 0, \ldots . V) = 0$ has simple roots only.*

And finally, differential equations which define the set of values of v'_1,

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* The extension of the entire discussion which follows to the case of multiple roots is too obvious to require special treatment. *Vid.* §1, 8, at end.

$v_2', \ldots v_n'$ which corresponds to any one of these values of V may be gotten by aid of (6) and (7). They are:

$$\left.\begin{aligned}
&x''\frac{dv_1'}{dx''}\left[v_1\frac{\partial V}{\partial v_1'}+V\right]+x''\frac{dv_2'}{dx''}v_1\frac{\partial V}{\partial v_2'}+\ldots+x''\frac{dv_n'}{dx''}v_1\frac{\partial V}{\partial v_n'},\\
&\qquad=-Vv_1'(r+s)-v_1x''\frac{\partial V}{\partial x''},\\
&x''\frac{dv_2'}{dx''}=\frac{s(v_1-v_2-v_1v_2)}{v_1},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&x''\frac{dv_n'}{dx''}=\frac{s(v_{n-1}-v_n-v_{n-1}v_n)}{v_{n-1}}.
\end{aligned}\right\}\qquad(9)$$

If in the first of these equations there be substituted for the products $x''\dfrac{dv_2'}{dx''}, x''\dfrac{dv_3'}{dx''}, \ldots$ the functions of the v_i''s to which the remaining equations make them equal, the set is reduced to a form from which the quantities $\left(\dfrac{dv_1'}{dx''}\right)_0, \left(\dfrac{dv_2'}{dx''}\right)_0, \ldots$ may be reckoned out by differentiating, setting $x''=0$, and solving the resultant set of linear equations; and from which the values of the higher differential coefficients of the v_i''s with respect to x'' for $x''=0$ may be reckoned out by successive repetitions of this process.

It will be supposed that the determinant of the coefficients in each of these sets of linear equations is different from zero; or (*vid.* this Journ. XI, 324) that

$$\begin{aligned}
&V_0(r+s+k)\quad(r+2s+k)\ldots(r+ns+k)\\
&+\left(\frac{\partial V}{\partial v_1'}\right)_0\frac{ks}{r+s}\quad(r+2s+k)\ldots(r+ns+k)\\
&+\left(\frac{\partial V}{\partial v_2'}\right)_0\frac{ks(r+s)}{r+2s}(r+3s+k)\ldots(r+ns+k)\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\\
&+\left(\frac{\partial V}{\partial v_n'}\right)_0\frac{ks(r+s)(r+2s)\ldots(r+(n-1)s)}{r+ns}
\end{aligned}$$

vanishes for no positive integral value of k. The process then leads to determinate series with finite coefficients; viz:

$$v_i'=\left(\frac{dv_i'}{dx''}\right)_0x''+\frac{1}{2!}\left(\frac{d^2v_i'}{dx''^2}\right)_0x''^2+\frac{1}{3!}\left(\frac{d^3v_i'}{dx''^3}\right)_0x''^3+\ldots\quad(i=1,2,\ldots n)$$

which formally satisfy the equations (9) and converge for regions of finite extent.

To obtain the development sought for—of y' in powers of x'—it then only remains to substitute $x'' = x'^{\frac{1}{s}}$ and the series thus obtained for the v_i''s in the equation

$$y' = Vv_1v_2 \ldots . v_n x'^n.$$

It will be holomorphic in $x'^{\frac{1}{s}}$ and have the form

$$y' = A_1 x'^{n+\frac{1}{s}} + A_2 x'^{n+\frac{2}{s}} + A_3 x'^{n+\frac{3}{s}} + \ldots .$$

The theory thus far developed leads directly to the following conclusions regarding the curves of the f-system which have the initial elements $x_0, y_0, y_1^0, \ldots y_n^0$ when for these elements $f = 0$ can be transformed into $F = 0$.

1. If $\left(\frac{\partial f}{\partial y_n}\right)_0 \neq 0$, there is but one group of lowest terms in $F = 0$, hence but one development for V in powers of x' and the variables v_i', and, therefore, speaking generally, but one development for y' in powers of x'; they will be integral powers, of course. In other words, there is but one curve of the system, defined by a differential equation $f(x, y, y_1, \ldots . y_n) = 0$, for each set of initial values of the variables, unless besides the equation $f = 0$ itself, these initial values satisfy the equation $\frac{\partial f}{\partial y_n} = 0$.

2. $\left(\frac{\partial f}{\partial y_n}\right)_0 = 0$, $\left(\frac{\partial f}{\partial y_{n-1}} y_n + \frac{\partial f}{\partial y_{n-2}} y_{n-1} + \ldots . + \frac{\partial f}{\partial y} y_1 + \frac{\partial f}{\partial x}\right)_0 \equiv \Delta f_0 \neq 0$.

The group of lowest terms in $F = 0$ then consists of the x' term—of which Δf_0 is the coefficient—and the $y_n'^2$ term; or if also

$$\left(\frac{\partial^2 f}{\partial y_n^2}\right)_0 = \left(\frac{\partial^3 f}{\partial y_n^3}\right)_0 = \ldots . \left(\frac{\partial^{s-1} f}{\partial y_n^{s-1}}\right)_0 = 0, \text{ but } \left(\frac{\partial^s f}{\partial y_n^s}\right)_0 \neq 0,$$

of the x' term and the $y_n'^s$ term.

For this group of terms the degree of y_n' in respect to x' is $\frac{1}{s}$; the corresponding V equation $\phi = 0$ (8) has s cyclicly connected roots each of which gives

rise finally to a development of the form

$$y' = A_1 x'^{n+\frac{1}{s}} + A_2 x'^{n+\frac{2}{s}} + A_3 x'^{n+\frac{3}{s}} + \dots .$$

In other words, s of the m curves which belong to an ordinary set of initial elements $x_0, y_0, \dots y^0_{n-1}$ and there have distinct y^0_n's are in the present case replaced by s branches of a single curve which are cyclicly connected and have the same y^0_n. At least two of the m curves will be replaced in this manner for every set of initial elements $x_0, y_0, \dots y^0_{n-1}$ which satisfy the equation $\mathrm{Disct}_{y_n} f = 0$, got by elimination of y_n from between $f = 0$ and $\dfrac{\partial f}{\partial y_n} = 0$; that is to say:

The differential equation of the $(n-1)^{th}$ order, $\mathrm{Disct}_{y_n} f = 0$, defines a system of curves each of which has at every one of its points a contact of order $n-1$ with a curve of the system $f = 0$ which there has a singularity of the cusp class characterized by the development:

$$y = y_0 + y^0_1 x' + y^0_2 \frac{x'^2}{2!} + \dots + y^0_n \frac{x'^n}{n!} + A x'^{n+\frac{1}{2}} + B x'^{n+1} + ^* \dots .$$

For an equation of the first order, as has already been noticed, this singularity is the ordinary cusp; for the equation of the second order, it is the cusp of the second species (Schnabelspitze), etc.

It need hardly be said that $\mathrm{Disct}_{y_n} f = 0$ will in general not satisfy the equation $f = 0$.

3. $\left(\dfrac{\partial f}{\partial y_n}\right)_0 = 0, \ \Delta f_0 = 0$.

The method of the preceding paragraphs ordinarily gives in this case two developments for y' in *integral* powers of x' both of the form

$$y' = A_1 x'^{n+1} + A_2 x'^{n+2} + A_3 x'^{n+3} + \dots .$$

developments of two curves of the system which have contact of the n^{th} order with each other.

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* This theorem is also given by Goursat, this Journ. XI, 4.

A full discussion of all the particular cases which suggest themselves at this point, however,—when other of the coefficients in $F=0$ besides those of y'_n and x' vanish—would mean the development of the entire theory of singular points (in the Poincaré sense) of differential equations of higher orders—which is quite aside from the purpose of the present paper.

We confine ourselves to the consideration of the hypothesis that Δf vanishes for *all* values of $x, y, \ldots . y_n$ which satisfy both the equations $f=0$ and $\frac{\partial f}{\partial y_n}=0$, or—if $\mathrm{Disct}_{y_n} f$ be resolvable into factors—for all values of x, $y, \ldots . y_{n-1}$ with the corresponding y_n—which satisfy the equation got by setting one of these factors equal to zero.

It is of course only in connection with special equations of the n^{th} order that this hypothesis is realized. Ordinarily when $f=0$, $\frac{\partial f}{\partial y_n}=0$ and $\Delta f=0$ together, the initial elements $x, y, \ldots . y_{n-2}$ satisfy an equation of the $(n-2)^{\text{th}}$ order got by eliminating y_n and y_{n-1} from between $f=0$, $\frac{\partial f}{\partial y_n}=0$ and $\Delta f=0$; but when our hypothesis is realized this eliminant equation vanishes identically—which means, of course, that f has a specialized form.

In this case $\mathrm{Disct}_{y_n} f=0$ satisfies the equation $f=0$, but in such manner that the curve system which it defines forms no part of the system defined by $f=0$. In other words, it is *a singular solution.*

The first statement is capable of a simple geometric demonstration; at least when additional coefficients of F do not vanish, viz:

Regard $x, y, \ldots . y_n$ as point coordinates in a flat space of n dimensions, S_n.

$y_n=0$ is then the equation of a flat space of $n-1$ dimensions, S_{n-1}, contained in S_n and in which $x, y, \ldots . y_{n-1}$ are point coordinates.

We thus gain two geometric pictures for the system of curves represented by the equation $f=0$.

1st. As a system of curves in the curved space $f(x, y, \ldots . y_n)=0$, one of which passes through each point $x, y, \ldots . y_n$ of this space.

2d. As a system of curves which is the projection of this first system in S_{n-1}, and of which m pass through each point $x, y, \ldots, y_{n-1}$ of this space.

Now the equation of the flat space of $n-1$ dimensions tangent to $f=0$ at any point $x_0, \ldots, y_n^0$ where $\frac{\partial f}{\partial y_n}=0$ is:

$$\left(\frac{\partial f}{\partial y_{n-1}}\right)_0 (y_{n-1}-y_{n-1}^0)+\ldots \left(\frac{\partial f}{\partial y}\right)_0 (y-y_0)+\left(\frac{\partial f}{\partial x}\right)_0 (x-x_0)=0,$$

which is also the equation of the S_{n-2} in which this tangent intersects S_{n-1}.

But as $x_0, y_0, \ldots y_n^0$ are given all values which satisfy the two equations $f=0$, $\frac{\partial f}{\partial y_n}=0$, this S_{n-2} envelopes the curved space of $n-2$ dimensions which is the picture of $\text{Disct}_{y_n} f=0$ in S_{n-1}.

It follows immediately that the curves of the system $\text{Disct}_{y_n} f=0$ satisfy the relation $\left(\frac{\partial f}{\partial y_{n-1}} y_n+\frac{\partial f}{\partial y_{n-2}} y_{n-1}+\ldots \frac{\partial f}{\partial y} y_1+\frac{\partial f}{\partial x}\right)_0=0$.

And since, by hypothesis, the curves of the system $f=0$ for whose initial elements $x_0, y_0, \ldots y_{n-1}^0$ the equation $\text{Disct}_{y_n} f=0$ holds good, satisfy this same relation; it is clear that the y_n^0 of the curve of $\text{Disct}_{y_n} f=0$ with the initial elements $x_0, y_0, \ldots y_{n-1}^0$ is the same as the y_n of a curve of $f=0$ with the same initial elements; or in other words, that the curves of $\text{Disct}_{y_n} f=0$ satisfy the equation $f=0$.

Therefore in the case now under consideration,

The differential equation of the $(n-1)^{th}$ order, $\text{Disct}_{y_n} f=0$, represents a system of curves each of which has at every one of its points contact of the n^{th} order with a curve of the system $f=0$.

But the thing most important to consider for its bearing on the theory of singular solutions is the degenerate form which the equations (9) take when Δf vanishes wherever f and $\frac{\partial f}{\partial y_n}$ vanish together. The consideration of this question leads also to the general demonstration that $\text{Disct}_{y_n} f=0$ then satisfies the equation $f=0$.

In effecting the transformation of $f=0$ into $\phi=0$, f was first transformed into F by means of the substitutions (2), or (4) and (5). It may readily be shown that the same substitutions transform $\frac{\partial f}{\partial y_n}$ into $\frac{\partial F}{\partial y_n'}$ and Δf into ΔF.

Second, by means of the substitutions (6) and (7), $F=0$ was transformed into $\phi(x'', v_1', v_2', \ldots . v_n', V)=0$. The corresponding transformations of $\frac{\partial F}{\partial y_n'}$ and ΔF are required.

To obtain them set out with the identity

$$F(x', y', \ldots . y_n') = X.\phi(x'', v_1', v_2', \ldots . v_n', V),$$

where X is an irrelevant factor of the form $x''^m v_1^{p_1} v_2^{p_2} \ldots . v_n^{p_n} V^q$ which may be common to all the terms of the complete expression in x'', v_1, ... V into which F is transformed by the foregoing substitutions.

Differentiating the two sides of this identity with respect to each of the quantities $v_1, \ldots . v_n$ we get the n identical equations:

$$\begin{aligned}\frac{\partial F}{\partial y'}\frac{Vv_1v_2\ldots . v_n x''^{r+ns}}{v_{n-p}} &+ \frac{\partial F}{\partial y_1'}\frac{Vv_1v_2\ldots . v_{n-1}x''^{r+(n-1)s}}{v_{n-p}} + \ldots . \\ &+ \frac{\partial F}{\partial y_p'}\frac{Vv_1v_2\ldots . v_{n-p}x''^{r+(n-p)s}}{v_{n-p}} \\ &= X\frac{\partial \phi}{\partial v_{n-p}} + \frac{\partial X}{\partial v_{n-p}}\phi;\ (p=0,\ 1,\ 2, \ldots . n-1).\end{aligned}$$

And differentiating with respect to V and x'', the additional equations:

$$\frac{\partial F}{\partial y'}v_1v_2\ldots v_n x''^{r+ns} + \frac{\partial F}{\partial y_1'}v_1v_2\ldots v_{n-1}x''^{r+(n-1)s} + \ldots + \frac{\partial F}{\partial y_n'}x''^{r} = X\frac{\partial \phi}{\partial V} + \frac{\partial X}{\partial V}\phi,$$

$$\begin{aligned}\frac{\partial F}{\partial y'}(r+ns)\,Vv_1v_2\ldots . v_n x''^{r+ns-1} &+ \frac{\partial F}{\partial y_1'}(r+(n-1)s)\,Vv_1v_2\ldots . v_{n-1}x''^{r+(n-1)s-1} \\ &+ \ldots . + \frac{\partial F}{\partial y_n'}\,r\,Vx''^{r-1} + \frac{\partial F}{\partial x'}\,s x''^{s-1} = X\frac{\partial \phi}{\partial x''} + \frac{\partial X}{\partial x''}\phi.\end{aligned}$$

By solving these equations for $\frac{\partial F}{\partial y'}, \frac{\partial F}{\partial y_1'}, \ldots . \frac{\partial F}{\partial y_n'}, \frac{\partial F}{\partial x'}$ and by a series of reductions which though somewhat tedious present no difficulties, we obtain the expressions which we are seeking, viz:

$$\left.\begin{aligned}\frac{\partial F}{\partial y_n'} &= \frac{X}{Vx''^{r}}\left[\frac{\partial \phi}{\partial V}V - \frac{\partial \phi}{\partial v_1}v_1\right] + Y_1\phi, \\ \Delta F &= \frac{X}{x''^{s}}\left[\frac{x''}{s}\frac{\partial \phi}{\partial x''} - \frac{r}{s}\frac{\partial \phi}{\partial V}V + \frac{\partial \phi}{\partial v_1}(1-v_1) + \frac{\partial \phi}{\partial v_2}\frac{v_1-v_2-v_1v_2}{v_1}\right. \\ &\quad \left. + \frac{\partial \phi}{\partial v}\frac{v_2-v_3-v_2v_3}{v_2} + \ldots . + \frac{\partial \phi}{\partial v_n}\frac{v_{n-1}-v_n-v_{n-1}v_n}{v_{n-1}}\right] + Y_2\phi.\end{aligned}\right\}(10)$$

Y_1 and Y_2 are linear functions of the irrelevant expressions $\frac{\partial X}{\partial v_i}, \frac{\partial X}{\partial V}, \frac{\partial X}{\partial x''}$.

It follows at once from (10) that when $F, \frac{\partial F}{\partial y'_n}$ and ΔF vanish together (for $x'' \gtrless 0$ as well as for $x'' = 0$), the same is true of ϕ, $\frac{\partial \phi}{\partial V} V - \frac{\partial \phi}{\partial v_1} v_1$ and

$$\frac{x''}{s}\frac{\partial \phi}{\partial x''} - \frac{r}{s}\frac{\partial \phi}{\partial V} V + \frac{\partial \phi}{\partial v_1}(1 - v_1) + \frac{\partial \phi}{\partial v_2}\,\frac{v_1 - v_2 - v_1 v_2}{v_1} + \frac{\partial \phi}{\partial v_3}\,\frac{v_2 - v_3 - v_2 v_3}{v_2} + \ldots + \frac{\partial \phi}{\partial v_n}\,\frac{v_{n-1} - v_n - v_{n-1} v_n}{v_{n-1}}.$$

In the light of these results consider the first of the equations of (9).

Since $\phi = 0$, $\frac{\partial \phi}{\partial x''} dx'' + \frac{\partial \phi}{\partial v_1} dv_1 + \frac{\partial \phi}{\partial v_2} dv_2 + \ldots + \frac{\partial \phi}{\partial v_n} dv_n + \frac{\partial \phi}{\partial V} dV = 0$;

whence
$$\frac{\partial V}{\partial v_1} = -\frac{\frac{\partial \phi}{\partial v_1}}{\frac{\partial \phi}{\partial V}}, \ldots \frac{\partial V}{\partial v_n} = -\frac{\frac{\partial \phi}{\partial v_n}}{\frac{\partial \phi}{\partial V}}, \quad \frac{\partial V}{\partial x''} = -\frac{\frac{\partial \phi}{\partial x''}}{\frac{\partial \phi}{\partial V}}.$$

The use of these equations and the last $(n-1)$ equations of (9) gives the first equation of this set the form:

$$x''\frac{dv'_1}{\partial x''}\left(\frac{\partial \phi}{\partial v_1} v_1 - \frac{\partial \phi}{\partial V} V\right) = (r+s)\frac{\partial \phi}{\partial V} V v_1 - \frac{\partial \phi}{\partial x''} v_1 x'' - s v_1\left[\frac{\partial \phi}{\partial v_2}\,\frac{v_1 - v_2 - v_1 v_2}{v_1} + \frac{\partial \phi}{\partial v_3}\,\frac{v_2 - v_3 - v_2 v_3}{v_2} + \ldots + \frac{\partial \phi}{\partial v_n}\,\frac{v_{n-1} - v_n - v_{n-1} v_n}{v_{n-1}}\right],$$

or when $s v_1\left(-\frac{x''}{s}\frac{\partial \phi}{\partial x''} + \frac{r}{s}\frac{\partial \phi}{\partial V} V - \frac{\partial \phi}{\partial v_1}(1 - v_1)\right)$ is added to the bracketed terms of its right number and at the same time subtracted from the terms without the brackets, the form:

$$x''\frac{dv'_1}{\partial x''}\left(\frac{\partial \phi}{\partial v_1} v_1 - \frac{\partial \phi}{\partial V} V\right) = s(v_1 - 1)\left(\frac{\partial \phi}{\partial v_1} v_1 - \frac{\partial \phi}{\partial V} V\right) + \left(\frac{x''}{s}\frac{\partial \phi}{\partial x''} - \frac{r}{s}\frac{\partial \phi}{\partial V} V + \frac{\partial \phi}{\partial v_1}(1 - v_1) + \frac{\partial \phi}{\partial v_2}\,\frac{v_1 - v_2 - v_1 v_2}{v_1} + \ldots + \frac{\partial \phi}{\partial v_n}\,\frac{v_{n-1} - v_n - v_{n-1} v_n}{v_{n-1}}\right).$$

From equations (10) it follows therefore, that in the case before us, this equation regarded as a differential equation is illusory. Values can, it is true, be found for the v_i''s which in an algebraic sense satisfy this and the remaining equations.

From either of the equations $\frac{\partial\phi}{\partial v_1} v_1 - \frac{\partial\phi}{\partial V} V = 0$, or

$$\frac{x''}{s}\frac{\partial\phi}{\partial x''} - \frac{r}{s}\frac{\partial\phi}{\partial V} V + \frac{\partial\phi}{\partial v_1}(1 - v_1) + \frac{\partial\phi}{\partial v_2}\frac{v_1 - v_2 - v_1 v_2}{v_1} + \ldots + \frac{v_{n-1} - v_n - v_{n-1}v_n}{v_{n-1}} = 0$$

a development may be had for v_1' in powers of x'' and $v_2', v_3', \ldots . v_n'$.

Let this value of v_1' be substituted in the equation

$$x''\frac{dv_2'}{dx''} = \frac{v_1 - v_2 - v_1 v_2}{v_1},$$

when it with the equations still remaining, viz.

$$x''\frac{dv_3'}{dx''} = \frac{v_2 - v_3 - v_2 v_3}{v_2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x''\frac{dv_n'}{dx''} = \frac{v_{n-1} - v_n - v_{n-1}v_n}{v_{n-1}},$$

will form a system from which series may be obtained for $v_2', v_3', \ldots . v_n'$ in powers of x'' by the method already described.

The substitution of these values of $v_2', v_3', \ldots . v_n'$ in the development already obtained for v_1' will give a series for it also in powers of x''.

But while these series may in an algebraic sense be called the solution of the equations (9), they are by no means such in the differential equation sense; no more indeed than the arbitrary function defined by the equation $\chi = 0$ is a solution of the differential equation $\frac{dy}{dx}\chi = Y\chi$.

And in the same sense and that only is the series to be gotten for y by the

substitution of these values of $v_1', v_2', \ldots . v_n'$ in the equation

$$y = y_0 + y_1^0 x' + y_2^0 \frac{x'^2}{2!} + \ldots . + y_n^0 \frac{x'^n}{n!} + Vv_1v_2 \ldots . v_n x'^{n+\frac{r}{s}},$$

a solution of the given equation $f = 0$.

We have thus arrived at a general criterion for singular solutions of the differential equation of any degree, which distinguishes them fully from ordinary or proper solutions and yet shows their place in the system of proper solutions; a criterion not based on geometrical considerations or assumptions as to the character of the curves defined by a differential equation, but based solely and directly on the differential equation itself.

A general equation $f = 0$ will have no singular solution; for it is a necessary condition for the occurrence of such a solution that Δf vanish wherever f and $\frac{\partial f}{\partial y_n}$ vanish together, and this imposes limitations on the coefficients of f.

When a singular solution exists, however, it will be met in applying the general methods which yield the solutions of $f = 0$—usually all proper solutions—which belong to a given set of initial elements $x_0, y_0, \ldots . y_n^0$. But in its case the first of the set of canonical equations to which $f = 0$ can be reduced for each of these solutions is satisfied not as a differential equation but only through the vanishing of factors algebraic in $x'', v_1', \ldots . v_n'$ which appear in all its terms.

§3.

Let $$\phi(x, y, c_1, c_2, \ldots . c_m) = 0$$

be any irreducible function of $x, y, c_1, c_2, \ldots . c_m$, one valued with respect to x, y for the region within which it is to be considered, and rational with respect to $c_1, c_2, \ldots . c_m$, where $c_1, c_2, \ldots . c_m$ satisfy $m - n$ algebraic equations $\psi_i(c_1, c_2, \ldots . c_m) = 0$, $(i = 1, \ldots . m - n)$, but are otherwise arbitrary.

$\phi = 0$ defines an n-ply infinite system of curves which satisfy a differential equation of order n, $f(y_n, y_{n-1}, \ldots . y, x) = 0$, the eliminant with respect to $c_1, c_2, \ldots . c_m$ of the $m + 1$ equations

$$\phi = 0, \ \frac{d\phi}{dx} = 0, \ \frac{d^2\phi}{dx^2} = 0, \ldots . , \frac{d^n\phi}{dx^n} = 0, \ \psi_i = 0 \qquad (i = 1, \ldots . m - n).$$

As $f=0$ depends solely on the manner in which $c_1, c_2, \ldots c_m$ are involved in $\phi=0$ and not at all on the character of these quantities, it is obvious that $\phi=0$ will continue to satisfy this equation when $c_1, c_2, \ldots c_m$ are replaced by any set of functions of x which satisfy the equations:

$$\sum_1^m \frac{\partial}{\partial c_i}(\phi)\frac{dc_i}{dx}=0,\quad \sum_1^m \frac{\partial}{\partial c_i}\left(\frac{d\phi}{dx}\right)\frac{dc_i}{dx}=0,\ \ldots \sum_1^m \frac{\partial}{\partial c_i}\left(\frac{d^{n-1}\phi}{dx^{n-1}}\right)\frac{dc_i}{dx}=0,$$

$$\sum_1^m \frac{\partial}{\partial c_i}(\psi_k)\frac{dc_i}{dx}=0 \qquad (k=1,\ \ldots m-n)$$

or the single equation

$$\Delta \equiv \begin{vmatrix} \dfrac{\partial\phi}{\partial c_1}, & \dfrac{\partial\phi}{\partial c_2}, & \ldots & \dfrac{\partial\phi}{\partial c_m} \\ \dfrac{\partial}{\partial c_1}\dfrac{d\phi}{dx}, & \dfrac{\partial}{\partial c_2}\dfrac{d\phi}{dx}, & \ldots & \dfrac{\partial}{\partial c_m}\dfrac{d\phi}{dx} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial}{\partial c_1}\dfrac{d^{n-1}\phi}{dx^{n-1}}, & \dfrac{\partial}{\partial c_2}\dfrac{d^{n-1}\phi}{dx^{n-1}}, & \ldots & \dfrac{\partial}{\partial c_m}\dfrac{d^{n-1}\phi}{dx^{n-1}} \\ \dfrac{\partial\psi_1}{\partial c_1}, & \dfrac{\partial\psi_1}{\partial c_2}, & \ldots & \dfrac{\partial\psi_1}{\partial c_m} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\psi_{m-n}}{\partial c_1}, & \dfrac{\partial\psi_{m-n}}{\partial c_2}, & \ldots & \dfrac{\partial\psi_{m-n}}{\partial c_m} \end{vmatrix} = 0.$$

The eliminant with respect to $c_1, c_2, \ldots c_m$ of the $m+1$ equations:

$$\phi=0,\ \frac{d\phi}{dx}=0,\ \ldots \frac{d^{n-1}\phi}{dx^{n-1}}=0,\ \ \psi_i=0,\quad \Delta=0,$$

a differential equation of order $n-1$,

$$\chi(y_{n-1}, y_{n-2}, \ldots y, x)=0,$$

will in general be a singular solution of the equation $f(y_n, \ldots y, x)=0$. Its solution will not be contained in the curve-system which $f=0$ defines, but it will satisfy this equation.

This is readily proven, as follows. Of the curves $\phi=0$ there is an $(n-1)$-

ply infinite system passing through any point x_0, y_0, the parameters of which satisfy the equations

$$\phi(x_0, y_0, c_1, \ldots c_m) = 0, \quad \psi_i(c_1, c_2, \ldots c_m) = 0, \qquad (i = 1, \ldots m - n).$$

Between the parameters $c_1, c_2, \ldots c_m$ of any curve of this system, and the parameters $c_1 + dc_1, \ldots c_m + dc_m$ of a second curve of the system consecutive to it, there must hold the relation:

$$\sum_1^m \frac{\partial\phi(x_0, y_0, c_1, \ldots c_m)}{\partial c_i} dc_i = 0 \ldots \qquad (1)$$

For the two curves to touch at x_0, y_0, it is further necessary that

$$\sum_1^m \frac{d}{dx}\left(\frac{\partial\phi(x, y, c_1, \ldots c_m)}{\partial c_i}\right)_{x_0, y_0} dc_i = 0 \ldots \qquad (2)$$

And for their contact to be of the order $n - 1$, still further that

$$\sum_1^m \frac{d^2}{dx^2}\left(\frac{\partial\phi(x, y, c_1, \ldots c_m)}{\partial c_i}\right)_{x_0, y_0} dc_i = 0 \ldots$$

$$\sum_1^m \frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial\phi(x, y, c_1, \ldots c_m)}{\partial c_i}\right)_{x_0, y_0} dc_i = 0 \ldots (3), \ldots (n);$$

the common $y_1, y_2, \ldots y_{n-1}$ of the curves being given by the equations

$$\left(\frac{d\phi}{dx}\right)_{x_0, y_0} = 0, \ldots, \left(\frac{d^{n-1}\phi}{dx^{n-1}}\right)_{x_0, y_0} = 0.$$

But the eliminant with respect to $dc_1, dc_2, \ldots dc_m$ of the equations $(1) \ldots (n)$ and the $m - n$ additional equations

$$\sum_1^m \frac{\partial\psi_k}{\partial c_i} dc_i = 0$$

is $\Delta = 0$, it having been supposed in the construction of the differential coefficients which constitute the elements of Δ as in the construction of those involved in the present eliminant, that $c_1, c_2, \ldots c_m$ are independent of x.

It follows immediately that $\chi(y_{n-1}, y_{n-2}, \ldots y, x) = 0$, the eliminant with respect to $c_1, c_2, \ldots c_m$ of the equations $\phi = 0, \dfrac{d\phi}{dx} = 0, \ldots \dfrac{d^{n-1}\phi}{dx^{n-1}} = 0,$

$\psi_i = 0$, and $\Delta = 0$, embodies the relation which must exist among the coordinates x, y of any point and the values at that point of the $y_1, y_2, \ldots . y_{n-1}$ of such consecutive curves of the system $\phi = 0$ as there have contact of the order $n - 1$ with each other.

A curve of the system $\chi = 0$ may therefore be described by moving a tracing point from $x_0 y_0$ to $x_1 y_1$ along an infinitesimal arc of $\phi_0 = 0$, a curve of the system $\phi = 0$; thence to $x_2 y_2$ along an infinitesimal arc of the curve $\phi_1 = 0$ which has a contact of the order $n - 1$ with $\phi_0 = 0$ at $x_0 y_0$; thence again to $x_3 y_3$ along an infinitesimal arc of $\phi_2 = 0$ which has contact of the order $n - 1$ with $\phi_1 = 0$ at $x_1 y_1$, etc. This curve, since it has the same $y_1, y_2, \ldots . y_{n-1}$ with $\phi_1 = 0$ at $x_0 y_0$ and also at the consecutive point $x_1 y_1$, has contact of the n^{th} order with $\phi_1 = 0$ at $x_0 y_0$; in like manner contact of the n^{th} order with $\phi_2 = 0$ at $x_1 y_1$, etc. It therefore satisfies the equation $f(y_n, y_{n-1}, \ldots . y, x) = 0$. Therefore

The equation $\chi = 0$ contains the envelope system of $\phi = 0$, a system of curves each of which has at each of its points contact of the n^{th} order with a curve of $\phi = 0$; and this system satisfies the equation $f = 0$.

It should be noticed, however, that $\chi = 0$ may contain systems of curves besides this envelope system—which do not satisfy the equation $f = 0$; may indeed contain such curves only.

By the hypothesis made at the outset there is a $1 - 1$ correspondence between the curves of $\phi = 0$ and the systems of values of the parameters $c_1, c_2, \ldots . c_m$. The equations $\phi = 0$, $\dfrac{d\phi}{dx} = 0, \ldots . \dfrac{d^{n-1}\phi}{dx^{n-1}} = 0$, $\psi_i = 0$, give $c_1, c_2, \ldots . c_m$ in terms of $x, y, y_1, \ldots . y_{n-1}$ and therefore determine the sets of values of these parameters which belong to that set of the curves $\phi = 0$ which pass through any point x, y, and there have given $y_1, y_2, \ldots . y_{n-1}$. In the general case these sets of values are distinct from one another, and so the curves of the set are distinct curves. But it may happen that two or more of the sets of values are the same. The corresponding curves then cannot be distinct, but must be but different branches of one and the same curve.

For the occurrence of loci of such singular points more conditions must be satisfied than there are independent parameters; ϕ must have a special form. But when they do occur, they satisfy the equation $\chi = 0$.

This may be readily seen by regarding $c_1, c_2, \ldots . c_m$ as the coordinates of a point in a space of m dimensions S_m.

$\phi = 0$, $\dfrac{d\phi}{dx} = 0$, ... $\dfrac{d^{n-1}\phi}{dx^{n-1}} = 0$, $\psi_i = 0$ are then the equations of m forms in this space, each of dimensionality $m - 1$, which have in common a finite number of points, generally distinct. If, however, two (or more) of these points coincide, the corresponding tangent S_{m-1}'s of the m forms have a line (S_1) in common, and the determinant Δ—whose elements are the coefficients of these tangent S_{m-1}'s—vanishes; that is, the condition $\chi = 0$ is satisfied.

Besides the envelope system, therefore, $\chi = 0$ includes the loci of points at which curves of the system $\phi = 0$ have the singularity of two curve branches having contact of the $n - 1^{th}$ order with each other, when such loci exist.

These loci, however, do not satisfy the equation $f = 0$, and hence constitute no part of its singular solution; for while $\chi = 0$ for the singular point itself on a curve $\phi_0 = 0$, this equation will generally not be satisfied for consecutive points of the curve, these points not being singular; or the values of y_n for $\chi = 0$ and $\phi = 0$ at the singular point are different.

PRINCETON COLLEGE, *July* 26, 1889.

On Confocal Bicircular Quartics.

By F. Franklin.

1. *Coordinates.* Throughout this paper the letters x, y will be understood to mean "circular coordinates," viz.

$$x = X + iY = re^{i\vartheta}, \quad y = X - iY = re^{-i\vartheta}, \tag{1}$$

X, Y being rectangular coordinates, and r, ϑ polar coordinates. A third variable z will, wherever convenient, be introduced for homogeneity, and then the definition of x and y will be understood to be modified so as to be given by

$$x : y : z = X + iY : X - iY : 1. \tag{2}$$

The points (x, z), (y, z) are the circular points I, J; the point (x, y) is the origin. It should be observed that a rotation of the axes X, Y through an angle α is equivalent to changing

$$x, y \quad \text{into} \quad e^{i\alpha}x, \ e^{-i\alpha}y. \tag{3}$$

2. *The axes of four concyclic points.* It will be desirable to obtain a formula relating to four concyclic points before taking up the consideration of bicircular quartics. If (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) be four points on a circle, the equality of the anharmonic ratios of the pencils through them from I and J respectively may be expressed by the equations

$$\frac{(x_1 - x_2)(x_3 - x_4)}{(y_1 - y_2)(y_3 - y_4)} = \frac{(x_1 - x_3)(x_2 - x_4)}{(y_1 - y_3)(y_2 - y_4)} = \frac{(x_1 - x_4)(x_2 - x_3)}{(y_1 - y_4)(y_2 - y_3)}. \tag{4}$$

Now the value of the first fraction is the clinant* of the pair of lines 12, 34; and likewise for the other fractions. In other words, if we denote by ϑ_{AB} the angle made with the axis of X by the line AB, the common value of the fractions is the value of $e^{2i(\vartheta_{12}+\vartheta_{34})}, = e^{2i(\vartheta_{13}+\vartheta_{24})}, = e^{2i(\vartheta_{14}+\vartheta_{23})}$.

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* See this Journal, XII, 162.

Or, what is the same thing, if α be the inclination (i. e. angle with axis of X) of an *axis* of the four points (i. e. of a line parallel to either bisector of any one of the pairs of lines (12, 34), (13, 24), (14, 23)), and if we denote $e^{2i\alpha}$ by Λ, the common value of the fractions is Λ^2.

Suppose, now, that the four points are given by the equations

$$Ax^4 + 4Bx^3z + 6Cx^2z^2 + 4Dxz^3 + Ez^4 = 0, \quad (5)$$

$$A'y^4 + 4B'y^3z + 6C'y^2z^2 + 4D'yz^3 + E'z^4 = 0; \quad (6)$$

the value of Λ^2 may be obtained as follows. Let

$$S = AE - 4BD + 3C^2, \quad T = ACE + 2BCD - AD^2 - EB^2 - C^3,$$
$$S' = A'E' - 4B'D' + 3C'^2, \quad T' = A'C'E' + 2B'C'D' - A'D'^2 - E'B'^2 - C'^3;$$

then, since S/A^2 is a homogeneous quadratic function and T/A^3 a homogeneous cubic function of the numerators of the fractions in (4), and since S'/A'^2 and T'/A'^3 are the like functions of the denominators, it follows that

$$\frac{A'^2S}{A^2S'} = (\Lambda^2)^2, \quad \frac{A'^3T}{A^3T'} = (\Lambda^2)^3,$$

whence Λ^2 is unambiguously determined. Since the pencils are homographic, we may suppose $S = S'$, $T = T'$; then

$$\Lambda^2 = \frac{A'}{A}.$$

It follows that when the axis of X is taken parallel to an axis of the four points, we have, in addition to $S = S'$ and $T = T'$, $A = A'$.

3. *Confocal bicircular quartics.* The general equation of a bicircular quartic may be written

$$kx^2y^2 + 2xyz(lx + my) + z^2(ax^2 + 2hxy + by^2) + 2z^3(gx + fy) + cz^4 = 0. \quad (7)$$

The system of tangents to this curve from the node (x, z) is found, on writing the equation in the form

$$(kx^2 + 2mxz + bz^2)y^2 + 2(lx^2 + hxz + fz^2)yz + (ax^2 + 2gxz + cz^2)z^2 = 0,$$

to be

$$(ak - l^2)x^4 + 2(am + gk - hl)x^3z + (ab - h^2 + ck - 2fl + 4gm)x^2z^2$$
$$+ 2(bg + cm - fh)xz^3 + (bc - f^2)z^4 = 0, \quad (8)$$

and likewise the system of tangents from the node (y, z) is

$$(bk - m^2)\,y^4 + 2\,(bl + fk - hm)\,y^3z + (ab - h^2 + ck - 2gm + 4fl)\,y^2z^2 \\ + 2\,(af + cl - gh)\,yz^3 + (ac - g^2)\,z^4 = 0\,. \quad (9)$$

These two pencils are known to be homographic; in point of fact, it is found on trial that the invariants of the quartics (8) and (9) are absolutely equal; and hence the 16 ordinary foci of the curve lie by fours on four circles.

To consider, now, a system of confocal bicircular quartics. The foci may be supposed to be given by the equations

$$(A,\, B,\, C,\, D,\, E\between x,\, z)^4 = 0\,, \quad (A',\, B',\, C',\, D',\, E'\between y,\, z)^4 = 0\,, \qquad (10)$$

and these quartics will be supposed so written that $S' = S$, $T' = T$. Then the conditions to which the 9 coefficients of the curve are subjected are obtained by equating the 10 coëfficients in (8) and (9) to the 10 coefficients in (10), each multiplied by a common multiplier λ; this gives, on the face of it, 10 equations homogeneous in 10 quantities. But since the coefficients in (8) and (9) satisfy *identically* the relations $S' = S$, $T' = T$, the number of effective equations is reduced to 8; hence there remains an arbitrary parameter; and therefore through any point in the plane there pass a finite number of curves belonging to the system.

How many curves pass through a given point may be determined by considering the curves through the origin, since no restriction has been made on the choice of origin. Putting, then, $c = 0$ and comparing (8) and (9) with (10), we have

$$f^2/g^2 = E/E'. \qquad (11)$$

But $gx + fy = 0$ is the tangent at the origin, and f/g is its clinant; hence there are *at most two directions** in which the curve may pass through the origin; and since the two values of f/g are negatives of each other, these two directions are mutually perpendicular.

If we write equation (11) in the form

$$\frac{A'}{A} \cdot \frac{f^2}{g^2} = \frac{E/A}{E'/A'}\,,$$

its full geometrical significance becomes evident. Viz. it is plain that

$$\frac{E}{A} \div \frac{E'}{A'}\,, = \frac{x_1x_2x_3x_4}{y_1y_2y_3y_4}\,,$$

*The above does not rigorously prove that there actually are two curves through every point, since it has not been shown that both values of f/g are admissible; it does prove that there are *at most* two curves through every point. The equation of the confocal system, which is obtained in the next article, of course determines the matter completely.

is the clinant of the system of rays drawn from the origin to the four foci; and it has been shown in art. 2 that A'/A is the square of the clinant of either axis of the four foci; hence equation (11) signifies that *the angle made with an axis of four concyclic foci of a bicircular quartic by the tangent to the quartic at any point is equal to half the sum of the angles made with the same axis by the four focal radii of the point.**

4. *Equation of the Confocal System.* The foci being given, as before, by equations (10), we may, without loss of geometrical generality, suppose $A' = A$ as well as $S' = S$ and $T' = T$, viz. this is equivalent (art. 3, end) to taking the axes of coordinates parallel to the axes of four concyclic foci. By a proper choice of origin we may effect a further simplification of the problem. Viz. the substitution of $x + \alpha z$ for x and $y + \beta z$ for y does not disturb the equalities already established; and it enables us to make $C' = C$ and $E' = E$. The five relations thus secured may be written as follows:

$$A = A',\quad C = C',\quad E = E',\quad BD = B'D',\quad A(D^2 - D'^2) + E(B^2 - B'^2) = 0.$$

The fourth of these equations will be satisfied if we put $B' = \rho B$ and $D = \rho D'$; or, slightly altering the notation, if we write instead of B, D, B', D' respectively $B, \rho D, \rho B, D$; and then the last equation becomes $(1 - \rho^2)(AD^2 - EB^2) = 0$. Thus the equations determining the foci become

$$Ax^4 + 4Bx^3z + 6Cx^2z^2 + 4\rho Dxz^3 + Ez^4 = 0, \tag{12}$$
$$Ay^4 + 4\rho By^3z + 6Cy^2z^2 + 4Dyz^3 + Ez^4 = 0, \tag{13}$$

with the relation among the coefficients

$$(1 - \rho^2)(AD^2 - EB^2) = 0. \tag{14}$$

It will be supposed throughout that neither A nor E vanishes; the vanishing of E would mean that the origin was a focus, the vanishing of A that one of the foci was at infinity.

We next observe that if $\rho = \pm 1$, or if B and D both vanish, four foci lie in a straight line, viz. it is obvious that four points satisfying equations (12) and (13) will in these cases lie either on $x = y$ or on $x = -y$. This is a special case which we shall at first exclude.

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* See "Note on the Double Periodicity of the Elliptic Functions," this Journal, XI, 285.

To find the equation of the confocal system, then, we have to subject the general equation of a bicircular quartic

$$kx^2y^2 + 2xyz(lx + my) + z^2(ax^2 + 2hxy + by^2) + 2z^3(gx + fy) + cz^4 = 0$$

to the conditions

$$\left.\begin{array}{lll} \quad(A) & \quad(B) & \quad(C) \\ ak - l^2 = A\lambda, & am + gk - hl = 2B\lambda, & ab - h^2 + ck - 2fl + 4gm = 6C\lambda, \\ bk - m^2 = A\lambda, & bl + fk - hm = 2\rho B\lambda, & ab - h^2 + ck - 2gm + 4fl = 6C\lambda, \\ & \quad(D) & \quad(E) \\ & bg + cm - fh = 2\rho D\lambda, & bc - f^2 = E\lambda, \\ & af + cl - gh = 2D\lambda, & ac - g^2 = E\lambda, \end{array}\right\} \quad (15)$$

where it is to be remembered that either $\rho^2 = 1$ or $AD^2 = EB^2$.

5. *The general case:* $\rho^2 \neq 1$, *B and D not both* 0. Since $AD^2 = EB^2$, and since $A \neq 0$ and $E \neq 0$, neither of the quantities B, D can vanish without the other; therefore neither of them vanishes. From equations (C) we get, by subtraction,

$$fl = gm; \qquad \text{i}$$

then from equations (D) and (E)

$$af^2 - bg^2 = \lambda E(b - a) = 2\lambda D(f - \rho g); \qquad \text{ii}$$

from equations (A) and (B)

$$am^2 - bl^2 = \lambda A(b - a) = 2\lambda B(m - \rho l); \qquad \text{iii}$$

from equations (A) $\qquad l^2 - m^2 = (a - b)k; \qquad$ iv

from equations (E) $\qquad f^2 - g^2 = (a - b)c; \qquad$ v

and from equations (B) and (D)

$$afm - bgl = 2\lambda B(f - \rho g) = 2\lambda D(m - \rho l). \qquad \text{vi}$$

We note that $a \neq b$; for if $b - a$ were 0, equations iv and v would give $l^2 - m^2 = 0$ and $f^2 - g^2 = 0$, while equations ii and iii would give $f - \rho g = 0$ and $m - \rho l = 0$; whence, since $\rho^2 \neq 1$, f, g, l, m must all vanish; but this is impossible, since equations (B) and (D) would then give $B = 0$, $D = 0$.

Since $b-a \neq 0$, it follows from ii and iii that $f-\rho g \neq 0$ and $m-\rho l \neq 0$; but, from i,
$$\frac{f}{m}=\frac{g}{l}=\frac{f-\rho g}{m-\rho l}, =\frac{D}{B} \text{ by vi;}$$
so that
$$f=\frac{D}{B}m, \quad g=\frac{D}{B}l. \qquad \text{vii}$$
By iii,
$$b-a=\frac{2B}{A}(m-\rho l), \qquad \text{viii}$$
while ii would give $b-a=\frac{2D}{E}(f-\rho g), =\frac{2D^2}{EB}(m-\rho l)$ by vii; and this is consistent with viii because $AD^2=EB^2$. Equations iv and v (in combination with vii and viii) give
$$k=\frac{A}{2B}\cdot\frac{m^2-l^2}{m-\rho l}, \quad c=\frac{E}{2B}\cdot\frac{m^2-l^2}{m-\rho l}. \qquad \text{ix}$$
From equations (B) we get
$$h=\frac{D}{B}k+\frac{bl-\rho am}{m-\rho l}, \qquad \text{x}$$
while equations D would give $h=\frac{B}{D}c+\frac{bg-\rho af}{f-\rho g}$, which is the same as the value in x because $c=\frac{E}{A}k=\frac{D^2}{B^2}k$ and $\frac{g}{f}=\frac{l}{m}$.

Equations vii, viii, ix and x, together with the equation
$$A(ab-h^2+ck+fl+gm)=3C(ak-l^2+bk-m^2); \qquad \text{xi}$$
[obtained from (A) and (C)] represent *all* of the given equations, the determination of λ being left out of account. Equations vii and ix express f, g, c, k in terms of l and m; it only remains, therefore, to determine a, b and h so as to satisfy viii, x and xi. Since $a-b=(l^2-m^2)/k$ by iv, we may put
$$a=\frac{l^2}{k}+t, \quad b=\frac{m^2}{k}+t,$$
(where it should be noticed that $t \neq 0$, otherwise either of equations (A) would give $A=0$) and then x gives
$$h=\frac{D}{B}k+\frac{lm}{k}+t\frac{l-\rho m}{m-\rho l}.$$

Hence

$$ab - h^2 = t^2\frac{(1-\rho^2)(m^2-l^2)}{(m-\rho l)^2}$$
$$+ t\left\{\frac{(m+\rho l)(m^2-l^2)}{k(m-\rho l)} - 2k\frac{D}{B}\frac{l-\rho m}{m-\rho l}\right\} - k^2\frac{D^2}{B^2} - 2lm\frac{D}{B}.$$

Also

$$k = \frac{A}{2B}\cdot\frac{m^2-l^2}{m-\rho l},\quad ck = \frac{D^2}{B^2}k^2,\ fl + gm = 2\frac{D}{B}lm, \text{ and } ak - l^2 = bk - m^2 = tk.$$

Making these substitutions, xi becomes

$$At\left\{t\frac{(1-\rho^2)(m^2-l^2)}{(m-\rho l)^2} + \frac{2B}{A}(m+\rho l) - \frac{AD}{B^2}\,\frac{(l-\rho m)(m^2-l^2)}{(m-\rho l)^2}\right\} = \frac{3AC}{B}\cdot\frac{m^2-l^2}{m-\rho l}\cdot t.$$

Rejecting the factor t, this equation gives

$$t = \frac{(m-\rho l)^2}{(1-\rho^2)(m^2-l^2)}\left\{\frac{3C}{B}\cdot\frac{m^2-l^2}{m-\rho l} + \frac{AD}{B^2}\cdot\frac{(l-\rho m)(m^2-l^2)}{(m-\rho l)^2} - \frac{2B}{A}(m+\rho l)\right\}.$$

Also

$$\frac{l^2}{k} = \frac{2B}{A}\cdot\frac{m-\rho l}{m^2-l^2}\cdot l^2 = \frac{(m-\rho l)^2}{(1-\rho^2)(m^2-l^2)}\cdot\frac{2B}{A}\cdot\frac{(1-\rho^2)\,l^2}{m-\rho l}.$$

Hence

$$a, = \frac{l^2}{k} + t,$$
$$= \frac{(m-\rho l)^2}{(1-\rho^2)(m^2-l^2)}\left\{\frac{3C}{B}\cdot\frac{m^2-l^2}{m-\rho l} + \frac{AD}{B^2}\cdot\frac{(l-\rho m)(m^2-l^2)}{(m-\rho l)^2} - \frac{2B}{A}\cdot\frac{m^2-l^2}{m-\rho l}\right\}$$

or $$(1-\rho^2)\,AB^2a = (3ABC - 2B^3)(m-\rho l) - A^2D\,(\rho m - l);$$

and likewise

$$(1-\rho^2)\,AB^2b = (3ABC - 2\rho^2B^3)(m-\rho l) - A^2D\,(\rho m - l).$$

It conduces to simplicity to express everything in terms of the new parameters α and β defined by

$$m - \rho l = (1-\rho^2)\,B\alpha,\quad \rho m - l = (1-\rho^2)\,B\beta;$$

whence $$l = B\,(\rho\alpha - \beta),\qquad m = B\,(\alpha - \rho\beta).$$

Then the above equations become

$$ABa = (3ABC - 2B^3)\,\alpha\ - A^2D\beta,$$
$$ABb = (3ABC - 2\rho^2B^3)\,\alpha - A^2D\beta;$$

equations vii and ix become

$$f = D(\alpha - \rho\beta), \quad g = D(\rho\alpha - \beta), \quad k = \frac{A}{2}\cdot\frac{\alpha^2 - \beta^2}{\alpha}, \quad c = \frac{E}{2}\cdot\frac{\alpha^2 - \beta^2}{\alpha};$$

and equation x gives

$$h = \frac{1}{2AB\alpha}\{A^2D(\alpha^2 + \beta^2) - 6ABC\alpha\beta + 4\rho B^3\alpha^2\}.$$

Hence, multiplying each of these values by $2AB\alpha$, and substituting in the equation

$$kx^2y^2 + 2xyz(lx + my) + z^2(ax^2 + 2hxy + by^2) + 2z^3(gx + fy) + cz^4 = 0,$$

we find the equation of the required curves to be

$$\begin{array}{lllll}
x^2y^2 \{ & A^2B\alpha^2 & & -A^2B\beta^2 & \} \\
+\,2x^2yz \{ & 2\rho AB^2\alpha^2 - & 2AB^2\alpha\beta & & \} \\
+\,2xy^2z \{ & 2AB^2\alpha^2 - & 2\rho AB^2\alpha\beta & & \} \\
+\,x^2z^2 \{ & (6ABC - 4B^3)\alpha^2 - & 2A^2D\alpha\beta & & \} \\
+\,2xyz^2 \{ & (A^2D + 4\rho B^3)\alpha^2 - & 6ABC\alpha\beta & + A^2D\beta^2 & \} \\
+\,y^2z^2 \{ & (6ABC - 4\rho^2B^3)\alpha^2 - & 2A^2D\alpha\beta & & \} \\
+\,2xz^3 \{ & 2\rho ABD\alpha^2 - & 2ABD\alpha\beta & & \} \\
+\,2yz^3 \{ & 2ABD\alpha^2 - & 2\rho ABD\alpha\beta & & \} \\
+\,z^4 \{ & ABE\alpha^2 & & -ABE\beta^2 & \} = 0.
\end{array} \qquad (16)$$

Let us call this

$$\alpha^2U - 2\alpha\beta V + \beta^2W = 0; \qquad (17)$$

then U and W are particular curves of the system (being obtained by putting α or $\beta = 0$), but V is not. The equation being quadratic in $\alpha : \beta$, there pass through every point in the plane two curves of the system.

In virtue of the relation $AD^2 = EB^2$, W is a perfect square; viz.

$$W = -A^2B\left(xy - \frac{D}{B}z^2\right)^2, \qquad (18)$$

and represents a circle with its centre at the origin, counted twice. And since it is obvious on inspection (bearing in mind that $AD^2 = EB^2$) that the relation $xy = \frac{D}{B}z^2$ converts the y-equation for the foci into the x-equation (equation (12) into equation (13)), this circle is a circle through four foci. Hence *each of the four focal circles, counted twice, is a curve of the system.*

If we put $\alpha = \pm\beta$ (and only so) the term in x^2y^2 disappears, and z becomes

a factor of the equation, so that we have as a limiting case of the quartics a circular cubic together with the line at infinity. But at the same time the term in z^4 disappears; hence the cubic passes through the origin, which has just been seen to be the centre of a focal circle. The terms of lowest degree in x and y are (dropping a constant factor)

$$x \mp y,$$

and the terms of highest degree are

$$xy(x \mp y).$$

Hence, there are two circular cubics confocal with the system of quartics; these cut each other orthogonally at the centre of each of the four focal circles; and the tangent to either cubic at each of these centres is parallel to the asymptote* of that cubic: the tangent and asymptote being in fact parallel to an axis of the foci.

6. *The case of four collinear foci.* If four foci are in a straight line, we may take this line as axis of X, and the centre of gravity of the four foci as origin; then the equations of the I-tangents and the J-tangents are identical: so that in equations (15) [end of art. 4] $\rho = 1$ and $B = 0$. Supposing $D \neq 0$, and noting that in obtaining equations i–vi of art. 5 the relation $AD^2 = EB^2$ (which does not hold in the case now under consideration) was not made use of, we see from iii that $a = b$; then from ii, $f = g$; and then from i, $l = m$. Hence the equations (15) for determining the curve reduce to

$$\left.\begin{aligned} ak - l^2 = A\lambda, \quad (a-h)\,l + fk = 0, \quad a^2 - h^2 + ck + 2fl = 6C\lambda, \\ (a-h)f + cl = 2D\lambda, \quad ac - f^2 = E\lambda. \end{aligned}\right\} (19)$$

Eliminating h by means of the second of these equations, the other four become

$$ak - l^2 = \quad A\lambda, \qquad \text{i}$$
$$ac - f^2 = \quad E\lambda, \qquad \text{ii}$$
$$-kf^2 + cl^2 = 2Dl\lambda, \qquad \text{iii}$$
$$-k^2f^2 - 2afkl + ckl^2 + 2fl^3 = 6Cl^2\lambda. \qquad \text{iv}$$

iii may be written $k(ac - f^2) - c(ak - l^2) = 2Dl\lambda$, or

$$Ac = Ek - 2Dl; \qquad \text{v.}$$

iv may be written $k(-kf^2 + cl^2) - 2fl(ak - l^2) = 6Cl^2\lambda$, or

$$Af = Dk - 3Cl; \qquad \text{vi}$$

*I. e. the asymptote other than the tangents at I and J; the *real* asymptote if the curve is real.

and i, ii, iii obviously give also

$$Af^2 - El^2 = -2Dla. \qquad \text{vii}$$

Equations v and vi give c and f in terms of k and l; substituting for f, equation vii becomes

$$2ADla = -D^2k^2 + 6CDkl + (AE - 9C^2)l^2; \qquad \text{viii}$$

and finally the second of equations (19) gives for h

$$2ADlh = 2ADla + 2ADfk = D^2k^2 + (AE - 9C^2)l^2. \qquad \text{ix}$$

Hence the equation of the required curves, which (since $a = b$, $f = g$, $l = m$) was

$$kx^2y^2 + 2l(x+y)xyz + [a(x^2+y^2) + 2hxy]z^2 + 2f(x+y)z^3 + cz^4 = 0$$

is found to be

$$\begin{array}{rl} x^2y^2\{ & 2ADkl \} \\ +\ 2(x+y)xyz\{ & 2ADl^2\} \\ +\ (x^2+y^2)z^2\{ & -D^2k^2 + 6CDkl + (AE - 9C^2)l^2\} \\ +\ 2xyz^2\{ & D^2k^2 + (AE - 9C^2)l^2\} \\ +\ 2(x+y)z^3\{ & 2D^2kl - 6CDl^2\} \\ +\ z^4\{ & 2DEkl - 4D^2l^2\} = 0, \end{array} \qquad (20)$$

or

$$-k^2D^2(x-y)^2z^2 + 2klD\{Ax^2y^2 + 3C(x^2+y^2)z^2 + 2D(x+y)z^3 + Ez^4\}$$
$$+ l^2\{4AD(x+y)xyz + (AE - 9C^2)(x+y)^2z^2 - 12CD(x+y)z^3 - 4D^2z^4\} = 0. \quad (21)$$

7. *Solution of the differential equation* $dx/\sqrt{Ax^4 + 4Bx^3 + 6Cx^2 + 4Dx + E} + dy/\sqrt{Ay^4 + 4By^3 + 6Cy^2 + 4Dy + E} = 0$. In virtue of the property of the tangent to a bicircular quartic contained in the theorem at the end of art. 3, the equation of the system of confocal bicircular quartics whose foci are given by

$$Ax^4 + 4Bx^3 + 6Cx^2 + 4Dx + E = 0, \quad Ay^4 + 4By^3 + 6Cy^2 + 4Dy + E = 0$$

is the solution of the differential equation

$$dx/\sqrt{Ax^4+4Bx^3+6Cx^2+4Dx+E} + dy/\sqrt{Ay^4+4By^3+6Cy^2+4Dy+E} = 0. \quad (22)$$

In the foregoing article we have obtained the equation of the system when the origin is so chosen that $B = 0$; so that the equation

$$dx'/\sqrt{Ax'^4 + 6C'x'^2 + 4D'x' + E'} + dy'/\sqrt{Ay'^4 + 6C'y'^2 + 4D'y' + E'} = 0 \quad (23)$$

has for its solution, by (21),

$$-k^2D'^2(x'-y')^2 + 2klD'\{Ax'^2y'^2 + 3C'(x'^2+y'^2) + 2D'(x'+y') + E'\}$$
$$+ l^2\{4AD'(x'+y')x'y' + (AE' - 9C'^2)(x'+y')^2 - 12C'D'(x'+y') - 4D'^2\} = 0, \quad (24)$$

or say $$-k^2D'^2(x'-y')^2+2klD'P+l^2Q=0. \tag{25}$$

To obtain the solution, then, of (22) we have to transform (25) [which is only an abbreviated expression of (24)] by the substitution
$$x'=x+\frac{B}{A},\quad y'=y+\frac{B}{A}, \tag{26}$$
expressing the result throughout in terms of the coefficients $A \ldots . E$. We shall write
$$u'=Ax'^4+6C'x'^2+4D'x'+E'=Ax^4+4Bx^3+6Cx^2+4Dx+E=u,$$
$$v'=Ay'^4+6C'y'^2+4D'y'+E'=Ay^4+4By^3+6Cy^2+4Dy+E=v.$$
In the first place, then, $x'-y'=x-y$. Secondly
$$\begin{aligned}2P&=u'+v'-A(x'^2-y'^2)^2\\&=u'+v'-A(x'-y')^2(x'+y')^2\\&=u+v-A(x-y)^2\Big(x+y+2\frac{B}{A}\Big)^2\\&=u+v-A(x^2-y^2)^2-4B(x^3+y^3)+4B(x+y)xy-4\frac{B^2}{A}(x-y)^2\end{aligned}$$
so that
$$P=Ax^2y^2+2B(x+y)xy+3C(x^2+y^2)+2D(x+y)+E-2\frac{B^2}{A}(x-y)^2. \tag{27}$$
It will, then, obviously be advantageous to put $kD'=\alpha-2\frac{B^2}{A}l$, α being a new arbitrary constant; whereupon equation (24) becomes
$$-\alpha^2(x-y)^2+2\alpha l\Big[P+2\frac{B^2}{A}(x-y)^2\Big]+l^2\Big[Q-4\frac{B^2}{A}P-4\frac{B^4}{A^2}(x-y)^2\Big]. \tag{28}$$
To transform Q it is convenient to observe that
$$\begin{aligned}\phi,\equiv 2(A^2D-3ABC+2B^3)x^3+(A^2E+2ABD-9AC^2+6B^2C)x^2\\+2(ABE-3ACD+2B^2D)x+EB^2-AD^2,\end{aligned}$$
is an x-covariant of $(A, B, C, D, E)(x, 1)^4$, i. e. a function which remains unaltered by the transformation $x=x'+\mu$; and the like expression in y will be denoted by ψ. For the quartic $(A, 0, C', D', E')(x', 1)^4$, ϕ becomes
$$A\,[2AD'x'^3+(AE'-9C'^2)x'^2-6C'D'x'-D'^2];$$
so that we have
$$\begin{aligned}A^2Q&=2A(\phi+\psi)-A^2(AE'-9C'^2)(x'-y')^2+4A^3D'(x'+y')x'y'-4A^3D'(x'^3+y'^3)\\&=2A(\phi+\psi)-A^2(x'-y')^2[4AD'(x'+y')+AE'-9C'^2]\\&=2A(\phi+\psi)-A^2(x-y)^2[4AD'(x+y)+AE'-9C'^2+8BD']\\&=2A(\phi+\psi)-(x-y)^2[4A(A^2D-3ABC+2B^3)(x+y)\\&\qquad+A^2(AE-4BD+3C^2)-12(AC-B^2)^2\\&\qquad+8B(A^2D-3ABC+2B^3)].\end{aligned} \tag{28}$$

Substituting for ϕ and ψ their values, and for P the value found in equation (27), we have

$$Q-4\frac{B^2}{A}P-4\frac{B^4}{A^2}(x-y)^2=-4B^2x^2y^2+(4AD-12BC)(x+y)xy$$
$$+(AE-9C^2)(x+y)^2+8BDxy+(4BE-12CD)(x+y)-4D^2,$$

so that the solution of (25) is

$$\begin{aligned}
&-\quad \alpha^2(x-y)^2\\
&+2\alpha l\,[Ax^2y^2+2B(x+y)xy+3C(x^2+y^2)+2D(x+y)+E]\\
&+\quad l^2\,[-4B^2x^2y^2+(4AD-12BC)(x+y)xy+(AE-9C^2)(x+y)^2\\
&\qquad\qquad +8BDxy+(4BE-12CD)(x+y)-4D^2]=0. \qquad (29)
\end{aligned}$$

[Cf. Cayley, Elliptic Functions, p. 339.]

8. *Case of four collinear foci symmetrically situated in respect to their centre of gravity; solution of the differential equation when B and D both vanish.* In art. 6, where the origin was taken so that $B=0$, it was expressly assumed that D was not also 0; and the equation there found [eq. (21), end of art. 6] ceases to represent a system of curves when in it we put $D=0$. It is, however, easy to write (21) in such a form that it shall continue to involve an arbitrary constant when $D=0$; it is also easy to investigate this case independently; but since we have just extended the solution in (21) so as to cover the general case (viz. that in which B is not supposed 0), it will be simplest to obtain the solution for the case $B=0$, $D=0$, from the solution for the general case, which is given by equation (29). We thus obtain, for the system of bicircular quartics whose foci are given by

$$Ax^4+6Cx^2+E=0,\quad Ay^4+6Cy^2+E=0, \qquad (30)$$

the equation

$$-\alpha^2(x-y)^2+2\alpha l\,[Ax^2y^2+3C(x^2+y^2)+E]+l^2(AE-9C^2)(x+y)^2=0. \qquad (31)$$

Putting $l\sqrt{AE-9C^2}=l'$, $\alpha/\sqrt{AE-9C^2}=\alpha'$, this may be otherwise written

$$-\alpha'^2(AE-9C^2)(x-y)^2+2\alpha'l'\,[Ax^2y^2+3C(x^2+y^2)+E]+l'^2(x+y)^2=0. \qquad (31')$$

When $AE-9C^2=0$, equation (31) may evidently be written

$$[Axy+3C+\mu(x-y)][Axy+3C-\mu(x-y)]=0, \qquad (32)$$

a pair of circles each passing through the two double foci given by $x=y$, $Axy+3C=0$; and equation (31′) may be written

$$[Axy-3C+\mu'(x+y)][Axy-3C-\mu'(x+y)]=0, \qquad (32')$$

a pair of circles through the two double foci given by $x = -y$, $Axy - 3C = 0$, and orthogonal to the preceding pair.

Equation (31) or (31′) is the solution of the differential equation

$$dx/\sqrt{Ax^4 + 6Cx^2 + E} + dy/\sqrt{Ay^4 + 6Cy^2 + E} = 0;$$

and in the particular case when $AE - 9C^2 = 0$, the two differential equations arising from this, viz.

$$dx/(Ax^2 + 3C) - dy/(Ay^2 + 3C) = 0, \quad dx/(Ax^2 + 3C) + dy/(Ay^2 + 3C) = 0,$$

have for their solution, by (32) and (32′),

$$Axy + 3C = \mu(x - y), \quad Axy - 3C = \mu'(x + y),$$

respectively.

9. *On the solution of the differential equation* $dx/\sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e}$ $= Mdy/\sqrt{ay^4 + 4by^3 + 6cy^2 + 4dy + e}$, *the quartics being homographic, and* M *being a certain constant.* Consider, first, the equation

$$\begin{aligned} &dx/\sqrt{Ax^4 + 4Bx^3 + 6Cx^2 + 4\rho Dx + E} \\ &= dy/\sqrt{Ay^4 + 4\rho By^3 + 6Cy^2 + 4Dy + E}. \qquad [AD^2 = EB^2] \end{aligned} \tag{33}$$

This defines a curve in which the inclination of the tangent is half the inclination of the system of rays drawn from its point of contact to a system of four concyclic points determined by

$$(A, B, C, \rho D, E \between x, 1)^4 = 0, \quad (A, \rho B, C, D, E \between y, 1)^4 = 0. \tag{34}$$

Hence the solution of (33) is the equation of the system of bicircular quartics whose foci are given by (34); so that this solution is furnished by equation (16), p. 330, z being therein replaced by 1.

Next, consider the more general equation

$$\begin{aligned} &dx/\sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} \\ &= dy/\sqrt{a'y^4 + 4b'y^3 + 6c'y^2 + 4d'y + e'}. \qquad [S' = S,\ T' = T] \end{aligned} \tag{35}$$

It defines a curve such that the square of the clinant of the tangent at any point is equal to the clinant of the system of rays drawn from the point to a system of four concyclic points determined by

$$(a, b, c, d, e \between x, 1)^4 = 0, \quad (a', b', c', d', e' \between y, 1)^4 = 0, \tag{36}$$

multiplied by a'/a. But, by art. 2, a'/a is the square of the clinant of an axis of these four points; hence, *with respect to an axis of the four points*, the inclina-

tion of the tangent is half the inclination of the system of rays drawn from its point of contact to the four points. The curve defined by (35) is therefore a bicircular quartic belonging to the confocal system whose foci are given by (36). By rotating the axes and moving the origin, i. e. by the transformation

$$x = e^{i\omega}(x_1 + \alpha), \quad y = e^{-i\omega}(y_1 + \beta), \tag{37}$$

equations (36) may be transformed (as shown at the beginning of art. 4) into equations (34); hence the equation of the confocal system just mentioned is given by (16), the x, y, z being replaced by x_1, y_1, 1. This result, converted from an expression in $A, \ldots, E, \rho, x_1, y_1$, into an expression in

$$a, \ldots, e, a', \ldots, e', x, y$$

would furnish the solution of (35).

Finally, if, in the equation

$$dx/\sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} = M dy/\sqrt{a'y^4 + 4b'y^3 + 6c'y^2 + 4d'y + e'},$$

the quartics are homographic, but it is not true that $S' = S$, $T' = T$, we may multiply the second quartic by q, when its invariants will become $S'' = q^2 S'$, $T'' = q^3 T'$; and we may choose q so that

$$q^2 S' = S, \quad q^3 T' = T,$$

whence $q = \dfrac{S'T}{ST'}$. The equation now becomes

$$\begin{aligned} &dx/\sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} \\ &= M\sqrt{\frac{S'T}{ST'}}\, dy/\sqrt{a''y^4 + 4b''y^3 + 6c''y^2 + 4d''y + e''}, \qquad [S'' = S,\ T'' = T] \end{aligned}$$

a'', b'', ... being written for qa', qb', ... Now, if $M = \sqrt{\dfrac{ST'}{S'T}}$, this equation becomes

$$\begin{aligned} &dx/\sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} \\ &= dy/\sqrt{a''y^4 + 4b''y^3 + 6c''y^2 + 4d''y + e''}. \qquad [S'' = S,\ T'' = T] \end{aligned}$$

Thus the solution of the equation

$$dx/\sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} = \sqrt{\frac{ST'}{S'T}}\, dy/\sqrt{a'y^4 + 4b'y^3 + 6c'y^2 + 4d'y + e'}, \tag{38}$$

the quartics being homographic, is at once reduced to that of equation (35).

On the Theory of Matrices.

By Henry Taber.

I.—Elements of the Theory.

Introductory.

§1. Cayley, in his *Memoir on the Theory of Matrices* (Phil. Trans., 1858), defined a matrix as "a set of quantities arranged in the form of a square,"* this notion arising "from an abbreviated notation for a set of linear equations." Accordingly, Cayley laid down the laws of combination of matrices upon the basis of the combined effect of the matrices as operators of linear transformation upon a set of scalar variables or carriers. The development of the theory, as contained in Cayley's memoir, was the development of the consequences of these primary laws of combination. Before Cayley's memoir appeared, Hamilton had investigated the theory of such a symbol of operation as would convert three vectors into three linear functions of those vectors, which he called a linear vector operator. Such an operator is essentially identical with a matrix as defined by Cayley; and some of the chief points in the theory of matrices were made out by Hamilton and published in his *Lectures on Quaternions* (1852). They were, however, made out as theorems concerning linear vector operators, and developed by quaternion methods, through the effect of these operators upon vectors, and not upon the basis of the primary laws of combination above referred to. Nevertheless, Hamilton must be regarded as the originator of the theory of matrices, as he was the first to show that the symbol of a linear transformation might be made the subject-matter of a calculus. Cayley makes no reference

* Cayley speaks also of *rectangular* matrices, and to some extent develops their theory; he even alludes to the possibility of attaching a more general meaning to the word. His memoir deals, however, almost exclusively with square matrices; and as the present paper relates exclusively to such, we shall make no further reference to other than square matrices.

to Hamilton, and was of course unaware that results essentially identical with some of his had been obtained by Hamilton; and, on the other hand, Hamilton's results related only to matrices of the third and fourth order, while Cayley's method was absolutely general. The identity of the two theories was explicitly mentioned by Clifford in a passage of his *Dynamic*, and was virtually recognized elsewhere by himself and by Tait. Sylvester carried the investigation much farther, developing the subject on the same basis as that which Cayley had adopted. Subsequent to Cayley, but previous to Sylvester, the Peirces, especially Charles Peirce, were led to the consideration of matrices from a different point of view; namely, from the investigation of linear associative algebras involving any number of linearly independent units. In this aspect, the quantities arranged in a square are looked upon as scalar coefficients of the several units or "vids" of an algebra belonging to a certain class. Finally, it must be mentioned that Hamilton and his successors made use of an interpretation of a linear vector operator which consists in regarding the operator or matrix as representing a homogeneous strain; and in this light the axes of the strain play an important part in the theory. These axes may, however, be utilized without any reference to this interpretation, the analytical definition of the axes being obvious; and I have made much use of them in the following investigation.

This paper originated in an investigation upon the development of Clifford's geometrical algebras; the consideration of the linear vector functions of these algebras led me to think of investigating the theory of matrices viewed as linear vector operators. For matrices in general these algebras furnish an instrument analogous to that which quaternions affords for the investigation of matrices of the third order. I shall, however, reserve to a subsequent paper the consideration of this particular system of algebras (of which I have obtained a tolerably complete development) and its utilization for the theory of matrices. In the present paper I regard a matrix as a linear unit operator, operating upon the linearly independent units of an algebra, without reference to any meaning of such units, or to any properties which these units may have in combination with each other; and I have in this way endeavored to develop the theory of matrices. From this point of view I am able to give a very simple proof of Cayley's "identical equation," and also of Sylvester's most important results, the law of latency, the law of nullity, and Sylvester's formula for any function of a single matrix. I have also completed the investigation of the nullity of the

factors of the identical equation (Sylvester's "corollary of the law of nullity") which Sylvester had treated only in the special case when all the latent roots of the matrix were distinct; and I have shown that in addition to nonions there are an infinity of algebras exactly analogous to quaternions. This analogy I have also extended, and it appears that every matrix is resolvable, precisely as a quaternion is, into the product of a tensor and a versor; the latter gives rise in a matrix of any order to functions analogous to the sine and cosine to which these functions reduce when $\omega = 2$; and thus I find that there exists an infinity of formulae analogous to De Moivre's. Finally, I have shown that the whole theory of matrices may be regarded as contained in the theory of Clifford's geometric algebras, i. e. in the theory of sets of quaternion algebras which are such that the units of one set are commutative with those of any other. Other results I have obtained not immediately connected with the pure theory of matrices, but having reference to the matrix viewed as a linear unit operator.

These results are contained in the second part of this paper; the first part contains only the elementary notions and theorems developed from the point of view of the matrix as an operator, and is not necessary to the understanding of the second part, at all events by one acquainted with Cayley's memoir. To this must be excepted §9, which contains an account of Charles Peirce's system of quadrate algebras and their connection with the theory of matrices.

I have in I §10 given a slight sketch of the history of the theory.

Definition of a Linear Unit Operator.

§2. The extension of any selection of ω of the units of any algebra will be termed a *ground* of order ω. A *linear unit operator* ϕ of a ground of order ω is an operator which converts each of ω linearly independent quantities in the ground into a quantity in the same ground, and which is such that

$$\phi(m\sigma + n\pi) = m(\phi\sigma) + n(\phi\pi),$$

where σ and π are any two quantities in the ground, and m and n are scalars.* Thus, if $\alpha_1, \alpha_2, \ldots \alpha_\omega$ are ω linearly independent quantities of a ground of order ω, and if

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_\omega\alpha_\omega,$$

* Of course this requirement is equivalent to the requirement that ϕ shall be distributive over any sum of quantities in the ground; the equation would then follow at once for m and n positive integers, and thence very simply for m and n any scalars.

then $$\phi\rho = x_1 . \phi\alpha_1 + x_2 . \phi\alpha_2 + \ldots . + x_\omega . \phi\alpha_\omega ;$$

where, since the $\phi\alpha$'s are in the same ground as the α's, and these are ω in number and linearly independent, we may put

$$\begin{aligned}
\phi\alpha_1 &= a_{11}\alpha_1 + a_{21}\alpha_2 + \ldots . + a_{\omega 1}\alpha_\omega , \\
\phi\alpha_2 &= a_{12}\alpha_1 + a_{22}\alpha_2 + \ldots . + a_{\omega 2}\alpha_\omega , \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\phi\alpha_\omega &= a_{1\omega}\alpha_1 + a_{2\omega}\alpha_2 + \ldots . + a_{\omega\omega}\alpha_\omega :
\end{aligned}$$

whence the *linear unit function* $\phi\rho$ becomes equal to

$$(a_{11}x_1 + a_{12}x_2 + \ldots . + a_{1\omega}x_\omega)\,\alpha_1 + (a_{21}x_1 + a_{22}x_2 + \ldots . + a_{2\omega}x_\omega)\,\alpha_2 + \ldots . \\ + (a_{\omega 1}x_1 + a_{\omega 2}x_2 + \ldots . + a_{\omega\omega}x_\omega)\,\alpha_\omega .$$

By choosing the α's as the units of the set $(x_1, x_2, \ldots . x_\omega)$, it may be regarded as identical with ρ, when it is evident that ϕ is the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots . & a_{1\omega} \\ a_{21} & a_{22} & \ldots . & a_{2\omega} \\ \ldots & \ldots & \ldots & \ldots \\ a_{\omega 1} & a_{\omega 2} & \ldots . & a_{\omega\omega} \end{pmatrix}$$

Thus, either the successive columns of the array constituting the matrix may be regarded as operating upon the successive quantities $(\alpha_1, \alpha_2, \ldots . \alpha_\omega)$ representing the ground, or the successive rows may be regarded as operating upon the coefficients $(x_1, x_2, \ldots . x_\omega)$ of these quantities.

Regarding ϕ as operating upon the set $\rho = (x_1, x_2, \ldots . x_\omega)$, i. e. upon the coefficients of the α's in the expression for ρ, the resulting set consisting of the coefficients in the expression for the linear unit function $\phi\rho$ may be denoted by $((\phi | x_1), (\phi | x_2), \ldots . (\phi | x_\omega))$; and so

$$\phi\rho = (\phi | x_1)\,\alpha_1 + (\phi | x_2)\,\alpha_2 + \ldots . + (\phi | x_\omega)\,\alpha_\omega .$$

In virtue of the identification of a matrix and a linear unit operator, these terms will hereafter be used interchangeably.

The coefficients of the α's in the expression for the $\phi\alpha$'s which form the columns of the matrix, will be termed its constituents.

The *converse* or *transverse* of ϕ, denoted by $\breve{\phi}$, will be defined as the matrix formed from ϕ by interchanging the rows and columns of its constituents.* Whence

* The term *transverse* was used by Cayley in his memoir to denote this function of a matrix, and has subsequently been used by Sylvester in the same sense; the term *converse* was that employed by Charles Peirce for this purpose, and the symbol of conversion or transversion employed here is also his. Hamilton used the term *conjugate* of ϕ to denote the converse of ϕ, and denoted it by ϕ'.

$$\breve{\phi}\rho = x_1 . \breve{\phi}\alpha_1 + x_2 . \breve{\phi}\alpha_2 + \ldots + x_\omega . \breve{\phi}\alpha_\omega ,$$

where
$$\begin{aligned} \breve{\phi}\alpha_1 &= a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots + a_{1\omega}\alpha_\omega , \\ \breve{\phi}\alpha_2 &= a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots + a_{2\omega}\alpha_\omega , \\ &\text{etc.} \end{aligned}$$

The relation "converse of" is evidently reciprocal, thus $\breve{\breve{\phi}} = \phi$.

Two matrices of the same ground are equal if they have invariably the same effect upon the same quantity. Thus if ψ is the matrix

$$\begin{pmatrix} b_{11} & b_{12} \ldots & b_{1\omega} \\ b_{21} & b_{22} \ldots & b_{2\omega} \\ \ldots & \ldots & \ldots \\ b_{\omega 1} & b_{\omega 2} \ldots & b_{\omega\omega} \end{pmatrix}$$

and the α's still represent the ground, so that

$$\begin{aligned} \psi\alpha_1 &= b_{11}\alpha_1 + b_{21}\alpha_2 + \ldots + b_{\omega 1}\alpha_\omega , \\ \psi\alpha_2 &= b_{12}\alpha_1 + b_{22}\alpha_2 + \ldots + b_{\omega 2}\alpha_\omega , \\ &\text{etc.,} \end{aligned}$$

and if $\phi\rho = \psi\rho$, for any quantity in the ground, then $a_{11} = b_{11}$, $a_{12} = b_{12}$, etc. Consequently, if two matrices are equal, every constituent of the one is equal to the corresponding constituent of the other; and so an equation between matrices gives rise to ω^2 scalar equations.

In defining the linear unit operator ϕ by its effect upon ω linearly independent quantities in a certain ground of order ω, the law of multiplication of the units of the ground is not in any way involved. Whence it follows that there is only an apparent loss of generality in taking the units of the ground from any particular algebra. If the elementary units of Clifford's ω-way algebra* are chosen, any quantity in the ground is a vector in ω-dimensional space; ϕ may then be termed a linear vector operator, and has a definite geometrical signification, namely, it represents a homogeneous strain. For if ρ and σ are vectors to two parallel straight lines, then we may put $\rho = \alpha_1 + x\beta$, $\sigma = \alpha_2 + y\beta$, and by definition $\phi\rho = \phi\alpha_1 + x\phi\beta$, $\phi\sigma = \phi\alpha_2 + y\phi\beta$, so the displacement of points in space which ϕ effects is such that parallel lines remain parallel; whence, extensions of any order which are parallel remain parallel after the application of the

* Clifford's ω-way algebra is an algebra linear in the product of even order of ω "elementary units," $i_1 i_2 \ldots i_\omega$, such that $i_r i_s = -i_s i_r$, and $i_r^2 = -1$. See this Journal, Vol. I.

strain ϕ. Without, however, determining the nature of the units of the ground, i. e. the properties its units may have in combination, two extensions belonging to the ground of order m and n respectively, where $m < n$, may be termed parallel if every quantity in the one may be represented, for some set of values of the x's, by

$$\alpha_1 + x_1\beta_1 + x_2\beta_2 + \ldots + x_m\beta_m,$$

and every quantity of the other, for some set of values of the y's, by

$$\alpha_2 + y_1\beta_1 + y_2\beta_2 + \ldots + y_m\beta_m + \ldots + y_n\beta_n;$$

the α's and β's being quantities of the ground. A homogeneous strain may then be defined as a displacement in a certain extension (namely, that of the ground) which leaves unaltered the parallelism of any two extensions of the ground. Obviously ϕ effects such a displacement of quantities.

A consequence of some interest follows from the identification of the theory of linear vector operators with the theory of matrices: as quaternions is identical with the theory of dual matrices, and thus with the theory of homogeneous strains in a plane, to every proposition concerning space of three dimensions (or of four dimensions) which can be proved by quaternions, corresponds a proposition concerning the kinematics of a plane, such that when either is proved so also is the other.

Hereafter it should be understood, that when quantities are spoken of as the operands of a matrix ϕ, they are to be regarded as in the ground pertaining to ϕ even when this is not explicitly stated.

Addition of Matrices, Multiplication by a Scalar, and Form of a Scalar.

§3. Employing the notation of the last section for ϕ and ψ, let χ denote the matrix

$$\left(\begin{array}{l} a_{11} + b_{11} \quad a_{12} + b_{12} \ldots . a_{1\omega} + b_{1\omega} \\ a_{21} + b_{21} \quad a_{22} + b_{22} \ldots . a_{2\omega} + b_{2\omega} \\ \text{etc., etc.} \end{array}\right)$$

It is obvious from the definition of a linear unit operator that for any quantity ρ,

$$\chi\rho = \phi\rho + \psi\rho.$$

Whence this sum may be denoted by $(\phi + \psi)\rho$, giving the equation in matrices, $\phi + \psi = \chi$, or

$$\begin{pmatrix} a_{11} & a_{12} \dots a_{1\omega} \\ a_{21} & a_{22} \dots a_{2\omega} \\ \text{etc.} & \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \dots b_{1\omega} \\ b_{21} & b_{22} \dots b_{2\omega} \\ \text{etc.} & \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \dots a_{1\omega}+b_{1\omega} \\ a_{21}+b_{21} & a_{22}+b_{22} \dots a_{2\omega}+b_{2\omega} \\ \text{etc.} & \end{pmatrix}$$

As a consequence of the nature of scalar addition, it follows that the addition of matrices is commutative and associative, and such that its inverse, subtraction, is determinative, or may be regarded as the direct operation performed with an inverse quantity, the negative. The negative of ϕ, which may be denoted by $(-\phi)$, is of course that matrix whose constituents are the negative of the corresponding constituents of ϕ; otherwise $\phi\rho + (-\phi)\rho$ would not be zero for all values of ρ.

Evidently the converse of the sum of two matrices is the sum of their converses. Thus:

$$\overbrace{(\phi + \psi)} = \breve{\phi} + \breve{\psi}.$$

§4. From the last section it follows that if m is any positive integer,

$$(m)\begin{pmatrix} a_{11} & a_{12} \dots a_{1\omega} \\ a_{21} & a_{22} \dots a_{2\omega} \\ \text{etc.} & \end{pmatrix} = \begin{pmatrix} ma_{11} & ma_{12} \dots ma_{1\omega} \\ ma_{21} & ma_{22} \dots ma_{2\omega} \\ \text{etc.} & \end{pmatrix}$$

Thence it may be shown very simply that the equation holds for any scalar m. And hence, obviously, for any scalar m, $(m\phi)\rho = \phi(m\rho) = m(\phi\rho)$. If, then, n is any other scalar, for any quantity ρ in the ground we have $(m.n\phi)\rho = m(n\phi.\rho) = m(n.\phi\rho) = mn.\phi\rho = (mn.\phi)\rho$; also $(m+n)\phi.\rho = (m+n).\phi\rho = m.\phi\rho + n.\phi\rho = m\phi.\rho + n\phi.\rho = (m\phi + n\phi)\rho$; consequently, $m.n\phi = mn.\phi$ and $(m+n)\phi = m\phi + n\phi$, and therefore scalars as multipliers of matrices have all the properties they possess in combination with monomial or scalar quantities. Hence in future the combination $m.\phi\rho = m\phi.\rho$ may be denoted by $m\phi\rho$, etc.

A scalar is itself obviously a linear unit operator or matrix; but the only matrix which merely multiplies a set or quantity by a scalar m is

$$\begin{pmatrix} m & 0 & 0 & \text{etc.} \\ 0 & m & 0 & \text{etc.} \\ 0 & 0 & m & \text{etc.} \\ \text{etc.} & & & \end{pmatrix}$$

where all the constituents are zero except those in the principal diagonal, and

they are all equal to m. If $m = 1$, this array gives the matrix form of unity; if $m = 0$, it gives the matrix form of zero. It will presently be shown that the product of two matrices is also a matrix, and that the product is distributive over the operand. In conjunction with the matrix form of a scalar, this makes significant the compound ϕm, m being a scalar. For since for any quantity ρ, $(\phi m)\rho = \phi(m\rho) = m(\phi\rho)$; hence $\phi m = m\phi$.

Multiplication of Matrices and its Inverse.

§5. The combination $\phi\psi\rho$ must be defined as the result of operating by the linear unit operator ϕ upon the linear vector function or quantity $\psi\rho$, i. e. $\phi\psi.\rho = \phi(\psi\rho)$. With the same notation as before for ϕ and ψ, let

$$\chi = \left(\begin{array}{cccc} \Sigma_r a_{1r} b_{r1} & \Sigma_r a_{1r} b_{r2} & \ldots & \Sigma_r a_{1r} b_{r\omega} \\ \Sigma_r a_{2r} b_{r1} & \Sigma_r a_{2r} b_{r2} & \ldots & \Sigma_r a_{2r} b_{r\omega} \\ \ldots & \ldots & \ldots & \ldots \\ \Sigma_r a_{\omega r} b_{r1} & \Sigma_r a_{\omega r} b_{r2} & \ldots & \Sigma_r a_{\omega r} b_{r\omega} \end{array}\right)$$

Then for any quantity ρ,

$$\begin{aligned} \phi\psi\rho &= \phi\,[\Sigma_s b_{1s} x_s . \alpha_1 + \Sigma_s b_{2s} x_s . \alpha_2 + \ldots + \Sigma_s b_{\omega s} x_s . \alpha_\omega] \\ &= (a_{11} . \Sigma_s b_{1s} x_s + a_{12} . \Sigma_s b_{2s} x_s + \ldots + a_{1\omega} . \Sigma_s b_{\omega s} x_s)\,\alpha_1 \\ &\quad + (a_{21} . \Sigma_s b_{1s} x_s + a_{22} . \Sigma_s b_{2s} x_s + \ldots + a_{2\omega} . \Sigma_s b_{\omega s} x_s)\,\alpha_2 + \text{etc.} \\ &= \Sigma_r \Sigma_s a_{1r} b_{rs} x_s . \alpha_1 + \Sigma_r \Sigma_s a_{2r} b_{rs} x_s . \alpha_2 + \ldots + \Sigma_r \Sigma_s a_{\omega r} b_{rs} x_s . \alpha_\omega \\ &= (\Sigma_r a_{1r} b_{r1} . x_1 + \Sigma_r a_{1r} b_{r2} . x_2 + \ldots + \Sigma_r a_{1r} b_{r\omega} . x_\omega)\,\alpha_1 \\ &\quad + (\Sigma_r a_{2r} b_{r1} . x_1 + \Sigma_r a_{2r} b_{r2} . x_2 + \ldots + \Sigma_r a_{2r} b_{r\omega} . x_\omega)\,\alpha_2 + \text{etc.} \\ &= \chi\rho . \end{aligned}$$

Whence we may put $\phi\psi = \chi$, or

$$\left(\begin{array}{cccc} a_{11} & a_{12} & \ldots & a_{1\omega} \\ a_{21} & a_{22} & \ldots & a_{2\omega} \\ \ldots & \ldots & \ldots & \ldots \\ a_{\omega 1} & a_{\omega 2} & \ldots & a_{\omega\omega} \end{array}\right)\left(\begin{array}{cccc} b_{11} & b_{12} & \ldots & b_{1\omega} \\ b_{21} & b_{22} & \ldots & b_{2\omega} \\ \ldots & \ldots & \ldots & \ldots \\ b_{\omega 1} & b_{\omega 2} & \ldots & b_{\omega\omega} \end{array}\right)$$

$$= \left(\begin{array}{lll} a_{11} b_{11} + a_{12} b_{21} + \ldots + a_{1\omega} b_{\omega 1}, & a_{11} b_{12} + a_{12} b_{22} + \ldots + a_{1\omega} b_{\omega 2}, & \text{etc.} \\ a_{21} b_{11} + a_{22} b_{21} + \ldots + a_{2\omega} b_{\omega 1}, & a_{21} b_{12} + a_{22} b_{22} + \ldots + a_{2\omega} b_{\omega 2}, & \text{etc.} \\ \ldots & \ldots & \ldots \\ a_{\omega 1} b_{11} + a_{\omega 2} b_{21} + \ldots + a_{\omega\omega} b_{\omega 1}, & a_{\omega 1} b_{12} + a_{\omega 2} b_{22} + \ldots + a_{\omega\omega} b_{\omega\omega}, & \text{etc.} \end{array}\right)$$

i. e. the product of the two matrices ϕ and ψ in the order named is formed as

follows: The constituent of the r^{th} row and s^{th} column of $\phi\psi$ is the sum of the products obtained by multiplying each constituent of the r^{th} row of ϕ into the corresponding constituent of the s^{th} column of ψ.

Since by definition the product of two matrices is associative with their operand, it follows that the multiplication of matrices is also associative. For by definition

$$\phi(\psi\chi).\rho = \phi.(\psi\chi)\rho = \phi(\psi.\chi\rho),$$

and likewise,

$$(\phi\psi)\chi.\rho = (\phi\psi).\chi\rho = \phi(\psi.\chi\rho).$$

As this is true for any quantity in the ground, hence

$$\phi(\psi\chi) = (\phi\psi)\chi.$$

The multiplication of matrices is also distributive over addition. For if the quantities ρ' and ρ'' denote severally the linear unit functions $\phi\rho$ and $\psi\rho$ of any quantity ρ in the ground, by §2 and §3,

$$\begin{aligned}\phi(\psi + \chi)\rho &= \phi(\psi\rho + \chi\rho) = \phi(\rho' + \rho'') \\ &= \phi\rho' + \phi\rho'' \quad = \phi\psi\rho + \phi\chi\rho \\ &= (\phi\psi + \phi\chi)\rho.\end{aligned}$$

The commutative principle does not in general hold in the calculus of matrices.

The converse of the product of two matrices is the product of their converses in the reverse order, as may readily be proved. This gives the formula

$$\phi\psi = \breve{\psi}\breve{\phi}.$$

§6. Division is the operation inverse to multiplication. Since multiplication is not in general commutative, two signs are required for division. In the last section, two matrices being given, it was required to find their product in either order; the problem inverse to this is to find a matrix which when multiplied into or by a given matrix ϕ shall have as product a given matrix ψ. In general, this problem is susceptible of one or more solutions: the matrix or matrices which when multiplied into ϕ give ψ as product may be denoted by $(\psi:\phi)$; and those which when multiplied by ϕ give ψ as product may be denoted by $\dfrac{\psi}{\phi}$ or $\dfrac{|\psi}{\phi|}$. Division is therefore defined by the equations

$$(\psi:\phi)\phi = \psi,$$

$$\phi\left(\frac{\psi}{\phi}\right) = \psi.$$

Invertible multiplication is multiplication whose inverse is determinative. If the multiplication of matrices is invertible, it would follow that if $\chi\phi=\psi$, and $\chi'\phi=\psi$, then $\chi=\chi'$; likewise, if $\phi\chi=\psi$ and $\phi\chi'=\psi$, then $\chi=\chi'$. Hence division would be subject further to the conditions

$$(\psi\phi):\phi=\psi,$$
$$\frac{\phi\psi}{\phi}=\psi,$$

by means of which the above results may be obtained immediately. To assume that multiplication is invertible is equivalent to regarding the inverse operation as the direct operation performed with an inverse quantity or matrix, which shares the associative and other properties of matrices in general. This inverse matrix is termed the reciprocal.*

It may now be shown that in general there exists a reciprocal, and hence that, in general, multiplication is invertible. From the rule for the formation of the product of two matrices, it follows that, if from the array of constituents representing a matrix ϕ another matrix Φ be formed, in which each constituent of the first array is replaced by the logarithmic differential derivative with respect to that constituent of the determinant of the array (provided this determinant is not null), the product of ϕ and the converse of Φ in either order is equal to unity.† Hence $\breve{\Phi}$ may be denoted by ϕ^{-1} and is termed the *reciprocal* of ϕ. In other words, if $|\phi|$ denote the determinant of the array representing the matrix (which will in future be termed the *content* of the matrix), and if Δ_ϕ represent the differential operator

$$\left(\begin{array}{cccc} \frac{\partial}{\partial a_{11}} & \frac{\partial}{\partial a_{12}} & \cdots & \frac{\partial}{\partial a_{1\omega}} \\ \frac{\partial}{\partial a_{21}} & \frac{\partial}{\partial a_{22}} & \cdots & \frac{\partial}{\partial a_{2\omega}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial}{\partial a_{\omega 1}} & \frac{\partial}{\partial a_{\omega 2}} & \cdots & \frac{\partial}{\partial a_{\omega\omega}} \end{array}\right)$$

*The term invertible multiplication was employed by Mr. C. S. Peirce in his *Logic of Relatives*, Memoirs Am. Acad., Vol. IX, in which, and in Mr. Peirce's other writings, will be found the substance of this account of division. Mr. Peirce has shown that the only algebras in which division is always determinative are ordinary algebra, with and without the imaginary, and real (semi) quaternions, this Journal, Vol. IV.

† In the wording of this statement of the relation between ϕ and Φ I have followed Prof. Sylvester, this Journal, Vol. VI.

then
$$\phi^{-1} = \frac{\breve{\Delta}_\phi |\phi|}{|\phi|}.$$

A matrix whose content is zero is termed *vacuous.* The inverse processes are sometimes possible with such a matrix; but since a vacuous matrix has no reciprocal, the results are indeterminate. Thus if

$$\varpi = \begin{pmatrix} a & 0 \\ ma & 0 \end{pmatrix},$$
$$\chi = \begin{pmatrix} b & 0 \\ mb & 0 \end{pmatrix},$$

the matrix $\left(\frac{\chi}{\varpi}\right)$ which when multiplied by ϖ gives χ as product, is

$$\begin{pmatrix} \frac{b}{a} & 0 \\ c & d \end{pmatrix}.$$

where c and d are any two scalars.

If $\phi\chi = 0$, it is evident that either both ϕ and ψ are vacuous, or one or both are zero.

From the definition of the reciprocal it follows that $(\breve{\phi})^{-1} = \breve{(\phi^{-1})}$. For taking the converse of both sides of the equation $\phi\phi^{-1} = 1$, we get $\breve{(\phi^{-1})}\breve{\phi} = 1$; hence $\breve{(\phi^{-1})} = (\breve{\phi})^{-1}$.

§7. Regarding a matrix as an operator, the problem inverse to that of finding the effect of a matrix ϕ upon any quantity ρ in the ground (i. e. of finding the product of ϕ into ρ) is, given a quantity σ in the ground, to find another quantity which ϕ will transform into σ. Only one sign is needed for the inverse of functional multiplication, which is defined by the equation

$$\phi\left(\frac{\sigma}{\phi}\right) = \sigma.$$

The quotient $\frac{\sigma}{\phi}$ is always determinate if ϕ is non-vacuous. If the matrix ϕ transforms any extension $(\alpha_1, \alpha_2, \ldots \alpha_m)$ into one of lower order $(\beta_1, \beta_2, \ldots \beta_n)$, and if $\sigma = y_1\beta_1 + y_2\beta_2 + \ldots + y_n\beta_n$, the problem to find a quantity which ϕ transforms into σ is in each case possible, but the quotient $\frac{\sigma}{\phi}$ is not in each case

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determinate. If $\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_m\alpha_m$, there does not exist in every case a quotient $\frac{\rho}{\phi}$.

Powers and Roots of a Matrix.

§8. If m is an integer, ϕ^m is of course that matrix which results from multiplying ϕ m times by itself. If n is also an integer, $\phi^{\frac{m}{n}}$ must be defined as that matrix whose n^{th} power is the m^{th} power of ϕ. The irrational scalar power ϕ^p of ϕ must be regarded, as in common algebra, as the limit of the series $\phi^{p'}$, $\phi^{p''}$, etc., where p', p'', etc., are successive approximations to p, i. e. if

$$\phi^{p'} = \begin{pmatrix} a'_{11} & a'_{12} & \text{etc.} \\ a'_{21} & a'_{22} & \text{etc.} \\ \text{etc.} & & \end{pmatrix}$$

and if the limits of the a''s as p' approaches the limit p are respectively the similarly situated constituents of the matrix

$$\psi = \begin{pmatrix} a_{11} & a_{12} & \text{etc.} \\ a_{21} & a_{22} & \text{etc.} \\ \text{etc.} & & \end{pmatrix},$$

then $\phi^p = \psi$. Whence $\phi^p - \phi^{p'}$ may be made as nearly equal to zero as we please by taking p' sufficiently near to p; and consequently, if $\phi^{p'}$ is susceptible of having a reciprocal, $\phi^p : \phi^{p'}$ may be made as near unity as we please by taking p' sufficiently near to p. It is obvious that if σ is the limit for any quantity ρ of $\phi^{p'}\rho$, as p' approaches the limit p, then $\phi^p\rho = \sigma$.

In general, the matrix ϕ^m where m is an integer has a reciprocal which may be denoted by $(\phi^m)^{-1}$. Prof. Sylvester shows, virtually as follows, that $(\phi^m)^{-1} = (\phi^{-1})^m$. Since $\phi\phi^{-1} = \phi^{-1}\phi = 1$, hence ϕ and ϕ^{-1} are commutative; consequently $1 = (\phi^{-1}\phi)^m = (\phi^{-1})^m\phi^m$, or $(\phi^m)^{-1} = (\phi^{-1})^m\phi^m(\phi^m)^{-1} = (\phi^{-1})^m$. Fractional and irrational powers of a matrix also have in general a reciprocal; and in this case also it may be shown that $(\phi^m)^{-1} = \phi^{-m}$. Whence it follows that for any two scalars $(\phi^m)^n = \phi^{(mn)}$.

It is obvious that $\phi^m\phi^n = \phi^{m+n}$ for any two integers m and n, being merely an expression of the associative principle; and it is sufficiently plain that with a proper understanding the equation holds for any two scalars.

Linear-form Representation of a Matrix.

§9. Let the set of forms or *vids**

$$\begin{array}{llll} (\alpha_1:\alpha_1) & (\alpha_1:\alpha_2) & (\alpha_1:\alpha_3) & \text{etc.,} \\ (\alpha_2:\alpha_1) & (\alpha_2:\alpha_2) & (\alpha_2:\alpha_3) & \text{etc.,} \\ (\alpha_3:\alpha_1) & (\alpha_3:\alpha_2) & (\alpha_3:\alpha_3) & \text{etc.,} \end{array}$$

as operators upon the quantities α_1, α_2, etc., be defined by the equations

$$(\alpha_r:\alpha_s)\,\alpha_s = \alpha_r,$$
$$(\alpha_r:\alpha_s)\,\alpha_t = 0;$$

and, when members of the set are operands, by

$$(\alpha_r:\alpha_s)\ \ (\alpha_s:\alpha_n) = (\alpha_r:\alpha_n),$$
$$(\alpha_r:\alpha_s)\ \ (\alpha_t:\alpha_n) = 0.$$

Then the expression

$$\begin{aligned} \Phi = a_{11}(\alpha_1:\alpha_1) &+ a_{12}(\alpha_1:\alpha_2) + a_{13}(\alpha_1:\alpha_3) + \text{etc.} \\ + a_{21}(\alpha_2:\alpha_1) &+ a_{22}(\alpha_2:\alpha_2) + a_{23}(\alpha_2:\alpha_3) + \text{etc.} \\ + a_{31}(\alpha_3:\alpha_1) &+ a_{32}(\alpha_3:\alpha_2) + a_{33}(\alpha_3:\alpha_3) + \text{etc.} + \text{etc.,} \end{aligned}$$

linear in these vids, considered as an operator upon any quantity

$$\rho = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + \text{etc.,}$$

is identical with the matrix

$$\phi = \left(\begin{array}{llll} a_{11} & a_{12} & a_{13} & \text{etc.} \\ a_{21} & a_{22} & a_{23} & \text{etc.} \\ a_{31} & a_{32} & a_{33} & \text{etc.} \\ \text{etc.} & & & \end{array}\right).$$

Moreover, if Ψ is a similar expression in the vids $(\alpha_r:\alpha_s)$ in which their coefficients the a's have been replaced by b's with corresponding subscripts, then $m\Phi + n\Psi$, where m and n are scalars, regarded as an operator upon ρ, is clearly identical with the matrix $m\phi + n\psi$. Likewise $\Phi\Psi$ has the same effect upon any operand ρ as the matrix $\phi\psi$; for

* The term *vid* was introduced by Mr. C. S. Peirce to denote the *units* or *letters* of an algebra. It will be employed in what follows to denote the forms $(\alpha_r:\alpha_s)$, which will also be termed the elementary units of a matrix.

$$\Phi\Psi = \Sigma_r\Sigma_s a_{rs}(\alpha_r : \alpha_s) . \Sigma_r\Sigma_s b_{rs}(\alpha_r : \alpha_s)$$
$$= \Sigma_r\Sigma_s\Sigma_t a_{rs} b_{st}(\alpha_r : \alpha_t),$$

which corresponds to the matrix

$$\left(\begin{array}{llll} \Sigma_r a_{1r} b_{r1} & \Sigma_r a_{1r} b_{r2} & \Sigma_r a_{1r} b_{r3} & \text{etc.} \\ \Sigma_r a_{2r} b_{r1} & \Sigma_r a_{2r} b_{r2} & \Sigma_r a_{2r} b_{r3} & \text{etc.} \\ \text{etc.} & & & \end{array}\right).$$

Any complete system of ω^2 of these vids forms a pure algebra of a certain class termed by Clifford *quadrates;* and expressed in terms of these units is in what may be termed its canonical form.* I shall therefore call an algebra linear in ω^2 of these vids a quadrate algebra of order ω; and any expression linear in the vids, a *quadrate form.* The multiplication tables to which these algebras give rise are similar, and are immediately obtained from the laws to which the vids are subject. Thus if $\omega = 2$, let

$$\begin{array}{cc} i & j \\ k & l \end{array}$$

denote a complete set of four of these vids. These letters or units give an algebra whose multiplication table is

	i	j	k	l
i	i	j	0	0
j	0	0	i	j
k	k	l	0	0
l	0	0	k	l

This is the algebra (g_4) of Prof. Peirce's linear associative algebras, and is a form of quaternions. If $\omega = 3$, let

$$\begin{array}{ccc} i & j & k \\ l & m & n \\ p & q & r \end{array}$$

denote the complete set of vids of the quadrate algebra of order three; these

*It seems appropriate to have a term to express that form of an algebra in which its units are capable of a classification, according to the requirements of the analysis of Peirce's *Linear Associative Algebra.* For this purpose I employ the term canonical form. Thus the form of quaternions given below is its canonical form; Hamilton's units are expressions linear in those given below.

letters give the algebra which was termed *nonions* by Clifford; its multiplication table is

	i	j	k	l	m	n	p	q	r
i	i	j	k	0	0	0	0	0	0
j	0	0	0	i	j	k	0	0	0
k	0	0	0	0	0	0	i	j	k
l	l	m	n	0	0	0	0	0	0
m	0	0	0	l	m	n	0	0	0
n	0	0	0	0	0	0	l	m	n
p	p	q	r	0	0	0	0	0	0
q	0	0	0	p	q	r	0	0	0
r	0	0	0	0	0	0	p	q	r

In virtue of the correspondence that has been shown in the first part of this section to exist between quadrate forms and matrices, it follows that to any function of one or more matrices corresponds a quadrate form which is the same function of one or more quadrate forms corresponding to the several matrices, and which has upon any operand ρ the same effect as the resulting matrix. Hence whatever equality subsists between combinations of matrices, if it can receive interpretation as like operations upon the ground, also subsists between the same combinations of corresponding quadrate forms. Thus it appears that there is no essential difference between the theory of matrices and the theory of quadrate forms. Viewed in this aspect, the scalar quantities arranged in a square and forming a matrix may be regarded as the scalar coefficients of the several vids. Or the substantial identity between the theory of matrices and of quadrate algebras may be brought out by considering all the possible ω^2 matrices of order ω which can be formed with one constituent unity and the remainder zero; when it is very readily seen that these matrices have the same multiplication table, and the same effect upon the ground as the ω^2 vids of the quadrate algebra of order ω; and since any matrix of order ω may be regarded as an expression in an algebra whose units are these ω^2 matrices, we thus have two algebras whose multiplication table is the same, and which, consequently, are identical. Hence a matrix is a quadrate form, and conversely. It is easily shown that the matrix of order ω whose non-zero constituent is in the r^{th} row and s^{th} column is identical

with the vid occupying the same place in the quadrate system of order ω. Thus if $\omega = 2$, the four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

in terms of which any dual matrix may be expressed, are respectively identical with the units i, j, k, l of the quadrate algebra of order four, or their equals the vids $(\alpha_1:\alpha_1)$, $(\alpha_1:\alpha_2)$, $(\alpha_2:\alpha_1)$, $(\alpha_2:\alpha_2)$; the matrices have, moreover, the same effect upon the ground (α_1, α_2) as these vids, etc.

The discovery of these systems of vids or of quadrate algebras is due to Mr. C. S. Peirce, by whom, through this discovery, the relation of the theory of quadrate forms (or the theory of matrices) to the general theory of multiple algebra was first made clear.

By §3,
$$\breve{\phi} = a_{11}\,\overbrace{(\alpha_1:\alpha_1)} + a_{12}\,\overbrace{(\alpha_1:\alpha_2)} + \text{etc.}$$
$$+ a_{21}\,\overbrace{(\alpha_2:\alpha_1)} + a_{22}\,\overbrace{(\alpha_2:\alpha_2)} + \text{etc.} + \text{etc.}$$

But by definition $\breve{\phi}$ is obtained from ϕ by interchanging its row and columns. Hence

$$\overbrace{(\alpha_r:\alpha_r)} = (\alpha_r:\alpha_r), \quad \overbrace{(\alpha_r:\alpha_s)} = (\alpha_s:\alpha_r).$$

The vids of the type $(\alpha_r:\alpha_r)$ I shall term self-transverse or self-converse vids; they may also be termed symmetric vids. Those of the type $(\alpha_r:\alpha_s)$ I shall term non-symmetric vids. Mr. Peirce terms the vids of the first type self-vids; the vids of the second type he terms alio-vids.

Sketch of the History of the Development of the Theory of Matrices.

§10. An outline of the origin of the theory of matrices was given in §1. Subsequent to Cayley's memoir, the next advance was made in 1870 by Charles S. Peirce, who, in his investigations upon the extension of the Boolian calculus to the logic of relatives,* came upon a set of forms (considered in §9) constituting a system virtually identical with the calculus of matrices. Peirce showed that any relative term involving not more than one correlate (dual relative)† could be represented as an expression linear in the units of a linear transformation.

* *Description of a Notation for the Logic of Relatives*, Memoirs Am. Acad. Sciences, Vol. IX (1870).
† Such as "lover of," "loved by," "mother of," etc.; but not "buyer of—from—," etc.

Whence follows the remarkable theorem that every such relation between any group of objects can be represented by a matrix. As has been stated, the relation of the theory of matrices, as algebras of a certain class (see §9), to linear associative algebra in general, was first made clear through the light thrown on the subject by Peirce's systems of vids.

Charles Peirce has made the great discovery that the whole theory of linear associative algebra is included in the theory of matrices. He has shown that every linear associative algebra has a relative form, i. e. its units may be expressed linearly in terms of the vids (denoted in his notation by $(A:A)$, $(A:B)$, etc.) of a linear transformation; and consequently, that any expression in the algebra can be represented by a matrix.* Whence the theory of all possible linear associative algebras is only the theory of all possible sets of matrices constituting a group in Benjamin Peirce's sense, i. e. which are such that the product of any two members of the set can be expressed linearly in terms of itself and the other members of the set alone. Charles Peirce has, moreover, given the relative or matrix form of all the algebras considered by his father in his *Linear Associative Algebra.*

To Charles Peirce, in conjunction with his father, the identification of quaternions with the quadrate algebra of order two (i. e. the algebra of dual matrices) is also due. Cayley, in his memoir, had remarked upon the similarity a certain system of three dual matrices had to the i, j, k of quaternion; but the identification was not completed until the remarkable discovery by Benjamin Peirce of a form of quaternions, which, in §9, I have termed the canonical form, and which results from choosing the linear functions

$$\frac{1+i\sqrt{-1}}{2},\quad \frac{j+k\sqrt{-1}}{2},\quad \frac{-j+k\sqrt{-1}}{2},\quad \frac{1-i\sqrt{-1}}{2},$$

of unity and any three mutually normal unit vectors i, j, k, as units of the algebra. These units obviously have the same multiplication table as the vids of a dual matrix. In his memoir of 1870 Charles Peirce had given, as an example of the infinite system of quadrate algebras, the multiplication table of the quadrate algebra next in order after quaternions, afterwards named *nonions* by Clifford; and he states in the Johns Hopkins Univ. Circs. No. 22 (April,

* See Mr. Peirce's *Logic of Relatives*, above referred to; also this Journal, Vol. IV, p. 221, and the Proc. Am. Acad. of Arts and Sciences for 1875.

1882), that the identification of quaternions with the quadrate algebra of order two suggested to his father and himself that, as nonions was thus shown to be the exact analogue of quaternions, there ought to be a form of nonions analogous to Hamilton's tetranomial form of quaternions, which form of nonions, Mr. Peirce states, either his father or himself found.* By means of this form any expression in the algebra is susceptible of representation linearly in terms of unity and eight non-scalar cube roots of unity, just as any expression in quaternions is susceptible of representation in terms of unity and three square roots of unity (or of -1). This form of nonions I shall term the octanomial form. The Peirces' discovery of the octanomial form of nonions was not published. The priority of publication of this form belongs to Sylvester, who discovered it subsequently to the Peirces without any knowledge of their investigations upon nonions.†

To Sylvester we owe most of the development of the theory of matrices. In his unfinished memoir in this Journal,‡ Sylvester distinguished between the different degree of vacuity and nullity of a matrix, replacing by these terms the term indeterminate used rather vaguely by Cayley to denote a null or vacuous matrix. It should be stated that Clifford had previously distinguished between the different degrees of nullity, employing the term indeterminate, with the prefix singly, doubly, etc. In this memoir Sylvester showed also how to derive the *chain of equations* from the identical equation, and the relation of these to the latent function of two or more matrices taken in a certain order. In a series of papers in the Johns Hopkins Univ. Circulars and in the Phil. Mag., Sylvester added largely to the theory of matrices.|| He demonstrated the extension of Hamilton's theorem concerning the cubic equation to which every matrix of the third order is subject, which was enunciated without proof by Cayley in 1858; among the other important results are what I term the second branch of the law of nullity, the corollary of the law of nullity, the law of latency, and the expression for any function of a single matrix. Sylvester has also extended the solution of

* "So much was published by me in 1870 [the multiplication table of the canonical form of nonions, etc.], and it then occurred to my father or to me (probably in conversing together) that since this algebra was thus shown (through his form of quaternions) to be the strict analogue of quaternions, there ought to exist a form of it analogous to Hamilton's standard tetranomial form. That form either he or I certainly found. I cannot remember which after so many years; whichever did must have found it at once."—*Johns Hopkins Univ. Circs.* No. 22 (April, 1882).

† Johns Hopkins Univ. Circs. Nos. 15 and 17 (1882).

‡ "Lectures on Universal Multiple Algebra," this Journal, Vol. VI (1883).

|| Nos. 15, 17, 27, 28, 32.

equations in quaternions or dual matrices beyond the point where Hamilton left the subject.

Clifford's contributions to the theory of matrices have been left to the last, though in point of time they preceded Sylvester's researches. They are contained in his "Fragment on Matrices," published in the posthumous edition of his works. I shall show subsequently that the main proposition of which he gives a proof is false. But the basis of his treatment of the subject is an important contribution to the theory of matrices. It is the method which is adopted in this paper; and the demonstration which I have given in the second part of this paper of the law of nullity is based on a result contained in Clifford's "Fragment."

II.—Development of the Theory by Means of the Axes of a Matrix.

Axes of a Matrix, or Quantities for which $(\phi - g)\rho = 0$, *where* g *is a Scalar.*

§1. If $\rho = \Sigma x_1 \alpha_1$ and $(\phi - g)\rho = 0$, then

$$(a_{11} - g)x_1 + a_{12}x_2 + \ldots + a_{1\omega}x_\omega = 0,$$
$$a_{21}x_1 + (a_{22} - g)x_2 + \ldots + a_{2\omega}x_\omega = 0,$$

etc., etc.

The resultant of this system is called the *latent function* of ϕ, and will be denoted by $|\phi - g|$, as it is the determinant of the matrix $\phi - g$. From its vanishing results an equation of order ω in g,

$$g^\omega - m_{\omega-1}g^{\omega-1} + m_{\omega-2}g^{\omega-2} - \ldots \mp m_1 g \pm m = 0.$$

The constant term m is evidently the determinant of the array of constituents forming the matrix, and has been denoted by $|\phi|$; m_1 is the sum of all the principal first minors of $|\phi|$; and, generally, $m_{\omega-\kappa}$ is the sum of all the principal κ^{th} minors of $|\phi|$. The term latent function is due to Prof. Sylvester, and the roots $g_1, g_2, \ldots g_\omega$ of the ω^{ic} he terms the *latent roots* of ϕ. If $|\phi| = 0$, it has already been stated that ϕ is then termed vacuous, and evidently has one latent root zero. If, in addition, $m_1 \neq 0$, ϕ has only one latent root zero, and is then said to have the vacuity one, or is simply vacuous. More generally, if all the m's from m to $m_{\kappa-1}$ are zero, and $m_\kappa \neq 0$, the matrix ϕ has just κ latent roots zero, and is said to have the vacuity κ. If $|\phi| \neq 0$, ϕ is non-vacuous, or its vacuity is zero.

Corresponding respectively to each latent root of ϕ are the ω quantities $\rho_1, \rho_2, \ldots \rho_\omega$, such that

$$0 = (\phi - g_1)\rho_1 = (\phi - g_2)\rho_2 = \ldots = (\phi - g_\omega)\rho_\omega.$$

These quantitities will be termed the *axes of* ϕ.

§2. There is no linear relation between any set of the axes of ϕ corresponding to distinct latent roots. For suppose any set of n axes are linearly related which correspond to the distinct latent roots $g_1, g_2, \ldots g_n$. Let $\varkappa$ be the number of these axes which are linearly independent, and suppose they correspond to the first $\varkappa$ latent roots. Then any axis corresponding to any of the remaining $n - \varkappa$ latent roots can be expressed linearly in terms of these. Thus

$$\rho_{\varkappa+1} = t_1\rho_1 + t_2\rho_2 + \ldots + t_\varkappa\rho_\varkappa,$$
$$\therefore\; g_{\varkappa+1}\rho_{\varkappa+1} = \phi\rho_{\varkappa+1} = t_1 g_1\rho_1 + t_2 g_2\rho_2 + \ldots + t_\varkappa g_\varkappa\rho_\varkappa;$$

but, since the t's are not all zero, this is impossible.

Whence, *if all the latent roots are distinct, the axes of* ϕ *are all linearly independent.* If a set of latent roots become equal, linear relations may arise between the set of axes corresponding to them, i. e. certain of these axes may be projected into the extension of the remaining axes corresponding to that set of latent roots, or all the axes of the set may become coincident.

If two or more axes of the set remain linearly independent when the set of latent roots become equal, these axes and also the remaining axes become indeterminate. Thus if the n latent roots $g_1, g_2, \ldots g_n$ ultimately become equal, of the axes corresponding to them only the first $\varkappa$ may remain linearly independent, and the remaining $n - \varkappa$ will then be expressible linearly in terms of them. These $\varkappa$ axes $\rho_1, \rho_2, \ldots \rho_\varkappa$ will all satisfy the equation $(\phi - g_1)\rho = 0$, and consequently any expression $x_1\rho_1 + x_2\rho_2 + \ldots + x_\varkappa\rho_\varkappa$ linear in them will also satisfy this equation, and hence will be an axis of ϕ. In this case any $\varkappa$ quantities giving the extension of $\rho_1, \rho_2, \ldots \rho_\varkappa$, together with any $n - \varkappa$ other quantities in their extension, may be regarded as the axes of ϕ corresponding to the n-fold latent root g_1. No quantity in the extension of $\rho_1, \rho_2, \ldots \rho_\varkappa$ can be in any linear relation with axes corresponding to latent roots other than g_1 by what has just been proved, as such quantities are axes of ϕ corresponding to g_1.

The matrix $\psi = \phi - g_1$ evidently has as latent roots $g_1 - g_1, g_2 - g_1, g_3 - g_1$, etc.; for if g is any latent root of ϕ, then $|\psi - (g - g_1)| = |(\phi - g_1) - (g - g_1)| = |\phi - g| = 0$; and hence $g - g_1$ is a latent root of ψ, since it is a root of the

latent function of ψ. Conversely, if $g-g_1$ is a latent root of ψ, g is a latent root of ϕ. Now if the axis of ϕ corresponding to g_1 (i. e. the axes of ψ corresponding to zero) is indeterminate, then every first minor of $|\phi-g_1|=|\psi|$ is zero; consequently every principal first minor of $|\psi|$ is zero; but then ψ has two latent roots zero, and consequently ϕ has two latent roots equal to g_1. Hence the axes corresponding to latent roots occurring only once among the latent roots of ϕ are never indeterminate.

Identical Equation.

§3. If ϕ is a matrix all of whose latent roots are distinct, any quantity ρ in the extension of the ground may be represented linearly in terms of its axes. Thus for any quantity ρ we may put

$$\rho=z_1\rho_1+z_2\rho_2+\ldots\ldots+z_\omega\rho_\omega.$$

Operating upon ρ by $\phi-g_\omega$ we get

$$(\phi-g_\omega)\rho=z_1(g_1-g_\omega)\rho_1+z_2(g_2-g_\omega)\rho_2+\ldots\ldots+z_\omega(g_\omega-g_\omega)\rho_\omega.$$

By this operation the last component of ρ, that along ρ_ω, is annulled. Operating on $(\phi-g_\omega)\rho$ by $\phi-g_{\omega-1}$, another component of ρ is annulled. Finally

$$(\phi-g_1)(\phi-g_2)\ldots\ldots(\phi-g_{\omega-1})(\phi-g_\omega)\rho=0.$$

This being true for any quantity ρ in the ground, may be written

$$(\phi-g_1)(\phi-g_2)\ldots\ldots(\phi-g_{\omega-1})(\phi-g_\omega)\equiv\phi^\omega-m_{\omega-1}\phi^{\omega-1}+\ldots\ldots\mp m_1\phi\pm m=0.$$

This is Cayley's "identical equation."

The proof of the identical equation may be extended to any case as follows: The latent roots being supposed to be all distinct, let $h_2=g_2-g_1$, $h_3=g_3-g_1$, etc., $h_n=g_n-g_1$, and let $\phi-g_1$ be denoted by ψ; finally, let χ denote the product $(\phi-g_{n+1})(\phi-g_{n+2})\ldots\ldots(\phi-g_\omega)$,—the identical equation then becomes

$$[\psi^n-\Sigma h_1\psi^{n-1}+\text{etc.}]\,\chi=\psi^n\chi-\Sigma h_1\psi^{n-1}\chi+\text{etc.}=0.$$

This equation is true however small the h's may be, provided the latent roots remain distinct. Consequently, as the h's diminish without limit, the first term of the second member $\psi^n\chi$ must ultimately diminish without limit, and in the limit the identical equation becomes

$$\psi^n\chi=(\phi-g_1)^n(\phi-g_{n+1})\ldots\ldots(\phi-g_\omega)=0.$$

Thus it is proved that the identical equation still subsists when any group of n latent roots becomes equal, the other latent roots being distinct. In the same way it may be shown that if any other set of the latent roots become equal, and finally whatever the relation between the latent roots, the identical equation subsists.

Cayley has remarked that the identical equation may be represented as follows:

$$\begin{vmatrix} a_{11}-\phi & a_{12} & \dots & a_{1\omega} \\ a_{21} & a_{22}-\phi & \dots & a_{2\omega} \\ \dots & \dots & \dots & \dots \\ a_{\omega 1} & a_{\omega 2} & \dots & a_{\omega\omega}-\phi \end{vmatrix} = 0 .$$

Or, as stated by Cayley, the determinant of the matrix less the matrix itself considered as involving the matrix unity is zero. This relation Cayley denotes symbolically by Det $(1\phi - \phi\tilde{1}) = 0$, where $\phi\tilde{1}$ signifies that ϕ is to be treated as a scalar. I propose to denote ϕ considered as involving the matrix unity by $\breve{\phi}$, when, with the notation previously employed for the determinant of a matrix, the identical equation may be represented by $|\phi - \breve{\phi}| = 0$.

Corresponding to any latent root g_1 of ϕ can, by §1, always be found an axis of ϕ; and upon this the effect of any operator $\phi - g_2$ for a latent root g_2 distinct from g_1 is by §2 only to multiply it by a scalar constant. Consequently no product of factors $\phi - g$, not containing $\phi - g_1$, can annul the axis ρ_1, and hence no such product can vanish. Similarly for every other latent root. Whence the identical equation must contain a factor $\phi - g$ for each distinct latent root. On the other hand, it was shown in §2 that when the latent roots are not all distinct, there may be more than one axis corresponding to a latent root g_1 which occurs more than once. In this case $\phi - g_1$ will annul an extension of order greater than unity, and the order of the identical equation will be lowered.

It is obvious that there can be but one identical equation, and that if ϕ is subject to any other equation involving only the matrix ϕ, this must contain the equation $|\phi - \breve{\phi}| = 0$ as a factor.

Converse of ϕ.

§4. The latent function $|\breve{\phi} - g|$ of $\breve{\phi}$ is obviously identical with the latent function of ϕ. Consequently the latent roots of ϕ and $\breve{\phi}$ are identical, and the identical equation in $\breve{\phi}$ is

$$(\breve{\phi} - g_1)(\breve{\phi} - g_2) \dots (\breve{\phi} - g_\omega) = 0 .$$

The axes of ϕ will be denoted by $\breve{\rho}_1, \breve{\rho}_2, \ldots . \breve{\rho}_\omega$. They do not necessarily coincide with those of ϕ. Indeed, I will show later that the axis $\breve{\rho}_1$ of $\breve{\phi}$ corresponding respectively to the latent root g_1 is normal to all the axes of ϕ, $\rho_2, \rho_3, \ldots \rho_\omega$ corresponding to the remaining latent roots.

On the Form Representing a Matrix.

§5. If the latent roots of ϕ are assigned, ϕ is subject to ω conditions. If in addition ω linearly independent axes are assigned to which the assigned latent roots are to correspond, ϕ is by definition completely determined. When the latent roots are given as not all distinct, and corresponding to them ω axes, of which those corresponding to a set of equal latent roots are not in every case linearly independent, ϕ is not completely determined; an infinity of matrices can be found which will have these latent root and corresponding axes, and the same identical equation.

To find the *form* or *array* representing the matrix ϕ whose latent roots g_1, g_2, etc., and corresponding axes ρ_1, ρ_2, etc., are assigned, we may proceed as follows. Let

$$\begin{aligned}
\rho_1 &= x_{11}\alpha_1 + x_{12}\alpha_2 + \ldots . + x_{1\omega}\alpha_\omega, \\
\rho_2 &= x_{21}\alpha_1 + x_{22}\alpha_2 + \ldots . + x_{2\omega}\alpha_\omega, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\rho_\omega &= x_{\omega 1}\alpha_1 + x_{\omega 2}\alpha_2 + \ldots . + x_{\omega\omega}\alpha_\omega .
\end{aligned}$$

Denoting by X the matrix formed from the array of the x's, evidently

$$\rho_1 = \breve{X}\alpha_1, \quad \rho_2 = \breve{X}\alpha_2, \quad \rho_3 = \breve{X}\alpha_3, \text{ etc.},$$

and by definition

$$\phi(\breve{X}\alpha_1) = g_1\breve{X}\alpha_1, \quad \phi(\breve{X}\alpha_2) = g_2\breve{X}\alpha_2, \quad \phi(\breve{X}\alpha_3) = g_3\breve{X}\alpha_3, \text{ etc.},$$

i. e. $$(\phi\between\breve{X}\between\alpha_1, \alpha_2, \ldots . \alpha_\omega) = (\breve{X}\between \begin{vmatrix} g_1 & 0 & \ldots . & 0 \\ 0 & g_2 & \ldots . & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots . & g_\omega \end{vmatrix} \between\alpha_1, \alpha_2, \ldots . \alpha_\omega)$$

As this equation is true for each of the α's, it is true for any expression linear in them, and so for any quantity in the ground. Hence

$$\phi = (\breve{X}\between \begin{vmatrix} g_1 & 0 & \ldots . & 0 \\ 0 & g_2 & \ldots . & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots . & g_\omega \end{vmatrix} \between\breve{X}^{-1})$$

This method is substantially identical with that given in Clifford's "Fragment on Matrices" by Cayley as a correction of Clifford's result.

If a set of ω linearly independent quantities other than the α's are chosen to represent the ground, the linear unit operator or matrix ϕ will, in general, be represented by another form or array. Two exceptions are, however, to be noted, the matrix unity and the matrix zero always have the same representation. By §2, if the latent roots of ϕ are all distinct, the axes of ϕ are capable of representing the ground. If they are chosen for this purpose, the form representing ϕ becomes

$$\left(\begin{array}{llll} g_1 & 0 & \ldots & 0 \\ 0 & g_2 & \ldots & 0 \\ \ldots & & & \\ 0 & 0 & \ldots & g_\omega \end{array}\right)$$

Evidently this form of a matrix gives an immediate proof of the identical equation. It has just been shown how from this form to pass to that form of ϕ when the α's are chosen to represent the ground, the axes being given in terms of the α's. If the α's are given in terms of the axes, and $\alpha_1 = \breve{X}\rho_1$, $\alpha_2 = \breve{X}\rho_2$, etc., the only change in the formula for ϕ is that $\breve{X}$ is to be replaced by $\breve{X}^{-1}$.

The more general problem, given the form representing a matrix ϕ in terms of certain ω linearly independent quantities (i. e. the form of ϕ, when these quantities are chosen to represent the ground), to find the form of ϕ in terms of another set of ω linearly independent quantities is solved in a similar way. Denote by ϕ_α the form of ϕ in terms of the set $(\alpha_1, \alpha_2, \ldots \alpha_\omega)$, and by ϕ_β the form of ϕ in terms of the set $(\beta_1, \beta_2, \ldots \beta_\omega)$, and let

$$\phi_\alpha = \left(\begin{array}{lll} a_{11} & a_{12} & \text{etc.} \\ a_{21} & a_{22} & \text{etc.} \\ \text{etc.} & & \end{array}\right) \qquad \phi_\beta = \left(\begin{array}{lll} b_{11} & b_{12} & \text{etc.} \\ b_{21} & b_{22} & \text{etc.} \\ \text{etc.} & & \end{array}\right)$$

If
$$\begin{aligned} \beta_1 &= x_{11}\alpha_1 + x_{12}\alpha_2 + \ldots + x_{1\omega}\alpha_\omega, \\ \beta_2 &= x_{21}\alpha_1 + x_{22}\alpha_2 + \ldots + x_{2\omega}\alpha_\omega, \\ &\ldots\ldots\ldots\ldots \\ \beta_\omega &= x_{\omega1}\alpha_1 + x_{\omega2}\alpha_2 + \ldots + x_{\omega\omega}\alpha_\omega, \end{aligned}$$

and X represents the matrix formed from the array of x's; then

$$\beta_1 = \breve{X}\alpha_1, \quad \beta_2 = \breve{X}\alpha_2, \quad \beta_3 = \breve{X}\alpha_3, \quad \text{etc.}$$

Operating by the matrix ϕ upon the β's and their equivalents gives

$$\phi_\alpha \breve{X} \alpha_1 = \phi_\beta \beta_1, \quad \phi_\alpha \breve{X} \alpha_2 = \phi_\beta \beta_2, \quad \phi_\alpha \breve{X} \alpha_3 = \phi_\beta \beta_3, \quad \text{etc.}$$

For, by definition, the effect of ϕ_α upon any expression in the α's is identical with the effect of ϕ_β upon the equivalent expression in the β's, since ϕ_α and ϕ_β both represent the same matrix. It is not proper yet to replace the β's by their expressions in terms of the α's, for ϕ_β as an operator upon the α's is not equivalent to the effect of the matrix ϕ. As the result of operating upon the β's by ϕ_β we have

$$\begin{aligned}
\phi_\alpha \breve{X} \alpha_1 &= b_{11}\beta_1 + b_{21}\beta_2 + \ldots + b_{\omega 1}\beta_\omega, \\
\phi_\alpha \breve{X} \alpha_2 &= b_{12}\beta_1 + b_{22}\beta_2 + \ldots + b_{\omega 2}\beta_\omega, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\phi_\alpha \breve{X} \alpha_\omega &= b_{1\omega}\beta_1 + b_{2\omega}\beta_2 + \ldots + b_{\omega\omega}\beta_\omega.
\end{aligned}$$

Replacing now the β's by their equivalents,

$$\begin{aligned}
\phi_\alpha \breve{X} \alpha_1 &= b_{11}\breve{X}\alpha_1 + b_{21}\breve{X}\alpha_2 + \ldots + b_{\omega 1}\breve{X}\alpha_\omega = \breve{X}\phi_\beta\alpha_1, \\
\phi_\alpha \breve{X} \alpha_2 &= b_{12}\breve{X}\alpha_1 + b_{22}\breve{X}\alpha_2 + \ldots + b_{\omega 2}\breve{X}\alpha_\omega = \breve{X}\phi_\beta\alpha_2, \\
&\text{etc.} \qquad \text{etc.}
\end{aligned}$$

i. e.
$$\phi_\alpha \breve{X}(\alpha_1, \alpha_2, \ldots \alpha_\omega) = \breve{X}\phi_\beta(\alpha_1, \alpha_2, \ldots \alpha_\omega),$$

where ϕ_β is now regarded as an operator upon the α's,

$$\therefore \ \phi_\alpha = \breve{X}\phi_\beta\breve{X}^{-1}.$$

§6. From the definition of a matrix as an operator upon a certain ground it follows that the identical equation does not vary with the form. It may, however, formally be proved as follows: Since $\phi_\alpha = \breve{X}\phi_\beta\breve{X}^{-1}$, evidently $\phi_\alpha^2 = \breve{X}\phi_\beta^2\breve{X}^{-1}$, etc., let the identical equation in ϕ_α be

$$\begin{aligned}
\phi_\alpha^\omega - m_{\omega-1}\phi_\alpha^{\omega-1} + \ldots \mp m_1\phi_\alpha \pm m &= 0, \\
\therefore \ \breve{X}(\phi_\beta^\omega - m_{\omega-1}\phi_\beta^{\omega-1} + \ldots \mp m_1\phi_\beta \pm m)\breve{X}^{-1} &= 0.
\end{aligned}$$

But since the β's constitute a set of ω linearly independent quantities, $|\breve{X}| \neq 0$. Consequently,

$$\phi_\beta^\omega - m_{\omega-1}\phi_\beta^{\omega-1} + \ldots \mp m_1\phi_\beta \pm m = 0.$$

Whence the vacuity of a matrix does not vary with its form. This proof of the invariance of the vacuity does not, however, take into consideration the fact that the identical equation may degrade. The invariance may, however, be shown as follows: If g is a scalar $\breve{X}g\breve{X}^{-1} = g$; hence

$$\phi_\alpha - g = \breve{X}\phi_\beta\breve{X}^{-1} - \breve{X}g\breve{X}^{-1} = \breve{X}(\phi_\beta - g)\breve{X}^{-1}.$$

From the definition of the product of two matrices it follows that the determinant of their product is equal to the product of their determinants.

$$\therefore\ |\phi_\alpha - g| = |\breve{X}|\,|\phi_\beta - g|\,|\breve{X}^{-1}| = |\phi_\beta - g|;$$

for $|\breve{X}^{-1}| = |\breve{X}|^{-1}$. Hence not only are the distinct latent roots of ϕ unchanged with its form, but the number of times each latent root is repeated is also unchanged.

Definition of Nullity and the Law of Nullity.

§7. A null matrix is one whose determinant vanishes, or of which all the minors of a certain order vanish. A non-null matrix is said to have a nullity zero, and one, every constituent of which is zero, is said to be absolutely null, or to have the nullity ω. It has been shown that the absolutely null matrix is the scalar quantity zero. Between these limits the number expressing the measure of nullity may have any integer value. If all the $(x-1)^{\text{th}}$ minors of the determinant of a matrix vanish, but not all the x^{th} minors, the matrix has a nullity x. Nullity of order one or simple nullity is evidently the same as simple vacuity. The vacuity of a matrix obviously cannot exceed its nullity, but it may have simple nullity and vacuity of any order from unity to ω.

The nullity of ϕ is not affected by multiplying it by a non-null matrix. Thus if the nullity of ϕ is m, the nullity of ϖ zero, the nullity of $\phi\varpi = \psi$ is m. For the $(m-1)^{\text{th}}$ minors of ψ consist of all possible products of a rectangular determinant formed from $\omega - m + 1$ rows of ϕ into the rectangular determinant formed from the corresponding $\omega - m + 1$ columns of ϖ; and each of these products is resolvable into the sum of $\omega - m + 1$ products of an $(m-1)^{\text{th}}$ minor of ϕ into an $(m-1)^{\text{th}}$ minor of ϖ. But the $(m-1)^{\text{th}}$ minors of ϕ are all zero. Whence the nullity of ψ is not less than m.

Since, however, $|\varpi| \neq 0$, $\phi = \psi\varpi^{-1}$, and in the same way it may be shown that the nullity of ϕ is not less than the nullity of ψ; hence the nullity of ψ is m. By the same method it may be shown that if $\varpi\phi = \psi$, the nullity of ψ is still equal to that of ϕ.* It follows immediately from this that the nullity of a matrix is unchanged when the form representing it is changed.

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*This proof follows the method employed by Sylvester to prove the theorem regarding the lower limit of the nullity of the product of two matrices. To Sylvester, as has been stated (I, §10), the term nullity and the discrimination between degrees of nullity are due.

§8. *If the greatest extension annulled by any matrix is m, its nullity is m, and conversely.* This I call the *first branch of the law of nullity.* Let $\alpha_1, \alpha_2, \ldots . \alpha_m$ be any m linearly independent quantities in the greatest extension A annulled by ϕ; and let $\beta_1, \beta_2, \ldots . \beta_n$ be any $n = \omega - m$ quantities in the extension B complementary to A with respect to the ground; the α's and β's may then be chosen to represent the ground, and any quantity ρ in the ground may be expressed by

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots . + x_m\alpha_m + y_1\beta_1 + y_2\beta_2 + \ldots . + y_n\beta_n,$$
$$\therefore \phi\rho = y_1\beta_1 + y_2\phi\beta_2 + \ldots . + y_n\phi\beta_n..$$

If
$$\begin{aligned}\phi\beta_1 &= a_{11}\alpha_1 + a_{21}\alpha_2 + \ldots . + a_{m1}\alpha_m + b_{11}\beta_1 + b_{21}\beta_2 + \ldots . + b_{n1}\beta_n,\\ \phi\beta_2 &= a_{12}\alpha_1 + a_{22}\alpha_2 + \ldots . + a_{m2}\alpha_m + b_{12}\beta_1 + b_{22}\beta_2 + \ldots . + b_{n2}\beta_n,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \phi\beta_n &= a_{1n}\alpha_1 + a_{2n}\alpha_2 + \ldots . + a_{mn}\alpha_m + b_{1n}\beta_1 + b_{2n}\beta_2 + \ldots . + b_{nn}\beta_n,\end{aligned}$$

then ϕ is represented by the form

$$\left|\begin{array}{c|ccc} & a_{11} & a_{12} & \ldots . a_{1n} \\ & a_{21} & a_{22} & \ldots . a_{2n} \\ & \ldots & \ldots & \ldots \\ & a_{m1} & a_{m2} & \ldots . a_{mn} \\ \hline & b_{11} & b_{12} & \ldots . b_{1n} \\ & b_{21} & b_{22} & \ldots . b_{2n} \\ & \ldots & \ldots & \ldots \\ & \ldots & \ldots & \ldots \\ & b_{n1} & b_{n2} & \ldots . b_{nn} \end{array}\right|$$

where the blank square and rectangle consist of m columns of zeros. Hence ϕ has a nullity at least m. If the nullity exceeds m, the determinant Δ formed by the n columns of the b's must vanish, and likewise every determinant which can be formed from $n = \omega - m$ rows of the last n columns of ϕ. Denoting the first minors of Δ, corresponding respectively to b_{11}, b_{12}, etc., by B_{11}, B_{12}, etc., we shall then have

$$\phi\,(B_{11}\beta_1 + B_{12}\beta_2 + \ldots . + B_{1n}\beta_n) = B_{11}\phi\beta_1 + B_{12}\phi\beta_2 + \ldots . + B_{1n}\phi\beta_n = 0.$$

But by supposition the extension A contains all the axes of ϕ corresponding to

the latent root zero.* Hence the nullity of ϕ cannot exceed m. The converse is immediately evident.

§9. *If the matrix ϕ transforms every quantity in the ground into one in an extension of order $\omega - m$, the nullity of ϕ is m, and conversely.* For if

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots\ldots + x_\omega\alpha_\omega$$

is any quantity in the ground, and

$$\phi\rho = (a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1\omega}x_\omega)\alpha_1 + (a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2\omega}x_\omega)\alpha_2 + \ldots\ldots + (a_{\omega 1}x_1 + a_{\omega 2}x_2 + \ldots\ldots + a_{1\omega}x_\omega)\alpha_\omega$$

is contained in an extension of order $\omega - m$; then there are m linear relations between the coefficients of the α's in the last equation true for all values of the x's. These give m sets of ω equations between the constituents of the matrix; namely,

$$\begin{aligned} l_1a_{11} + l_2a_{21} + \ldots\ldots + l_\omega a_{\omega 1} &= 0, \\ l_1a_{12} + l_2a_{22} + \ldots\ldots + l_\omega a_{\omega 2} &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_1a_\omega + l_2a_{2\omega} + \ldots\ldots + l_\omega a_{\omega\omega} &= 0, \end{aligned}$$

and $m - 1$ other sets of ω equations, in which the l's are replaced in each set by other constants. From the m equations obtained by taking the first equation in each set, $m - 1$ of the a's may be eliminated, leaving one equation between any $\omega - m + 1$ of the constituents of the first column; and, similarly, an equation linear in the corresponding constituents of the second column may be obtained from the second equations of the m sets, etc. Thus result ω equations linear in the constituents of each column taken from any $\omega - m + 1$ rows of $|\phi|$, the coefficients of the a's being the same for all constituents in the same row. The resultants of all sets of $\omega - m + 1$ of these equations, together with the like resultants formed from all such groups of ω equations, constitute the $(m-1)^{\text{th}}$ minors of $|\phi|$; and these resultants must be zero. Whence the proposition. The converse is evident.

If, as in the preceding section, A denote the total extension annulled by ϕ, and B the complementary extension, it will not in general happen that B is the

* If not only Δ but also all the B's vanish, the analysis needs correction. But, since the constituents in the last n columns of ϕ do not all vanish, the minors of order p (for some value p from $p = 1$ to $p = n - 1$) do not all vanish; and then in a similar way, if the minors of order greater than p all vanish, it may be shown that there would result a linear relation between $\phi\beta_1$, $\phi\beta_2$, etc., i. e. ϕ would annul a certain expression linear in the β's.

extension into which ϕ transforms any quantity of the ground. As will appear later, this is only true when the vacuity of ϕ does not exceed its nullity. Let B_1 denote that part of B transformed by ϕ into A. There must exist such an extension unless no extension but A is annulled by any power of ϕ. For if $\phi^{\kappa+1}$ (where $\kappa \neq 0$) is the lowest power of ϕ that annuls an extension C other than A, then $\phi^{\kappa} C$ is included in A, since by supposition A is the total extension annulled by ϕ; moreover, $\phi^{\kappa-1} C$ is not included in A, since it is not annulled by ϕ; hence $\phi^{\kappa-1} C$ is a part of the ground complementary to A that is transformed by ϕ into A. Let B_2 denote that part of the extension $B - B_1$ which is transformed by ϕ into B_1. By an argument similar to that above, it may be shown that if the order of B_2 is zero, then no power of ϕ annuls an extension other than the aggregate of A and B_1, i. e. $A + B_1$; etc. Finally, let B_p denote that part of the extension $B - (B_1 + B_2 + \ldots. + B_{p-1})$ which is transformed by ϕ into B_{p-1}; and suppose no portion of the remaining extension

$$B_{p+1} = B - (B_1 + B_2 + \ldots. + B_{p-1} + B_p)$$

is transformed by ϕ into B_p. Then all the extensions $A, B_1, B_2, \ldots. B_{p-1}, B_p$ are annulled by some power of ϕ, but no part of the extension B_{p+1} complementary to their aggregate is annulled by any power of ϕ. The following multiplication table shows the effect of ϕ upon these mutually exclusive extensions:

	A	B_1	B_2	etc.	B_{p-1}	B_p	B_{p+1}
ϕ	0	A'	B'_1	....	B'_{p-2}	B'_{p-1}	B'_{p+1}
ϕ^2	0	0	A''	....	B''_{p-3}	B''_{p-2}	B''_{p+1}
ϕ^3	0	0	0	....	B'''_{p-4}	B'''_{p-3}	B'''_{p+1}
etc.	etc.	....	....	....	....	....	....
ϕ^{p-1}	0	0	0		$A^{(p-1)}$	$B_1^{(p-1)}$	$B_{p+1}^{(p-1)}$
ϕ^p	0	0	0	.	0	$A^{(p)}$	$B_{p+1}^{(p)}$

In this table the accented letters denote extensions included each in the extensions denoted by the same letter with fewer accents or unaccented.

The total extension A of order m annulled by ϕ I shall term the null extension or *null region* of the matrix, and the aggregate of the extensions exclusive of A annulled by some power of ϕ the vacuous extension or *vacuous region* of the matrix. The extension or region into which ϕ project any quantity of the ground is

$$\phi(A + B_1 + B_2 + \ldots + B_{p-1} + B_p + B_{p+1}) \\ = A' + B_1' + B_2' + \ldots + B_{p-1}' + B_{p+1}';$$

and this I shall term the *residual region* of ϕ: If $\{A'\}$ denote the order of the extension A', etc., evidently

$$\{A'\} \leqq \{B_1\}, \quad \{B_1'\} \leqq \{B_2\} \text{ etc. } \{B_{p-1}'\} \leqq \{B_p\}, \quad \{B_{p+1}'\} \leqq \{B_{p+1}\},^*$$
$$\therefore \{A'\} + \sum_1^{p-1} {}_\kappa \{B_\kappa'\} + \{B_{p+1}'\} \leqq \sum_1^{p+1} {}_\kappa \{B_\kappa\} = \omega - m.$$

But by the proposition just proved the order of the residual extension is $\omega - m$,

$$\therefore \{A'\} = \{B_1\}, \quad \{B_1'\} = \{B_2\}, \text{ etc. } \{B_{p-1}'\} = \{B_p\}, \quad \{B_{p+1}'\} = \{B_{p+1}\},$$

and so $B_{p+1}' = B_{p+1}$.

Hence if C is any extension which has no part in common with the null region of ϕ, it is transformed by ϕ into an extension of the same order.

§10. The second branch of the law of nullity is easily derived from the two preceding sections. Denoting as before by A the null region of ϕ of nullity m, and by B the extension complementary to A with respect to the ground, let C denote the null-region of ψ of nullity n, and let $\mathfrak{B}$ denote that part of B which ϕ transforms into C. Obviously, the null-region of $\psi\phi$ is the aggregate of A, the region which ϕ annuls, and of $\mathfrak{B}$, that part of the ground complementary to A which ϕ transforms into C. By what was proved in §9 it follows that the order of $\phi\mathfrak{B}$ is the same as that of $\mathfrak{B}$. Whence to determine the order of $\mathfrak{B}$ it suffices to determine the order of $\phi\mathfrak{B}$, the extension common to C and the residual region of ϕ. The order of the residual region of ϕ is $\omega - m$ and the order of C is n; hence the extension $\phi\mathfrak{B}$ common to C and the residual region of ϕ

* It is obvious from the definition of a linear unit operator that it cannot increase the order of an extension; so if $\phi A = B$, hence $\{B\} \leqq \{A\}$.

is at most of order n. If $(\omega - m) + n > \omega$, i. e. $m < n$, the extension C and the residual region of ϕ have common an extension of order at least $((\omega - m) + n) - \omega = n - m$, but if $(\omega - m) + n \leqq \omega$, i. e. $m \geqq n$, these two extensions do not necessarily have any part common. Hence the order of $\phi\mathfrak{B}$, and consequently of $\mathfrak{B}$, is at most n; and if $m < n$, the order of $\mathfrak{B}$ is not less than $n - m$, but if $m \geqq n$, the order of $\mathfrak{B}$ may be zero. The extension A is of order m and has no part in common with $\mathfrak{B}$. Hence the null-region of $\psi\phi$ cannot be of order greater than $m + n$; and if $m < n$, it cannot be less than $(n - m) + m = n$, but if $m \geqq n$, it cannot be less than m; and, consequently, the nullity of *the product of two matrices is not greater than the sum of their nullities nor less than the greater nullity of the two matrices.** This theorem is due to Prof. Sylvester, who terms it the law of nullity. Owing to the importance of the relation of the null-extension to the nullity of a matrix, I term the whole relation of the null-extension and nullity of one or more matrices the law of nullity, and this theorem the second branch of the law.

Suppose that C, the null-region of ψ, has no part in common with the vacuous region of ϕ; let E denote the extension common to C and A, the null region of ϕ; then the residue of C, namely $C - E$, is wholly contained in that part of the ground which is complementary to the aggregate of the null-region of ϕ and the vacuous region of ϕ, and which was denoted in §9 by B_{p+1}. Let $\mathfrak{B}_1$ denote that part of the ground complementary to A which is transformed by ϕ into E ($\mathfrak{B}_1$ is evidently contained in B_1); let the order of E be p and the order of $\mathfrak{B}_1$ be q. Since $\phi B_{p+1} = B_{p+1}$ (§9), the null-region of $\psi\phi$ is obviously the aggregate A, of $\mathfrak{B}_1$, and of $C - E$, which are all mutually exclusive; and, consequently the order of the null-region of $\psi\phi$ is $m + q + (n - p)$. But $p \leqq m$, $p \leqq n$; moreover, the order of the extension $\phi\mathfrak{B}_1$ is equal to that of $\mathfrak{B}_1$, but $\phi\mathfrak{B}_1$ is included in E, hence $q \leqq p$. If $p = 0$, the order of null-region of $\psi\phi$ is $m + n$. In this case the null-region of ψ is included wholly in that part of the ground complementary to the aggregate of the null-region of ϕ and the vacuous region of ψ. The aggregate of these two extensions I shall term, for reasons which

* The outline of this proof was communicated by me to the Johns Hopkins University Mathematical Society in a paper read before the Society, Nov. 1888, when I was informed by Dr. Franklin that this method of proving Sylvester's law had been employed by Mr. Bucheim in a note in the Phil. Mag. (XVIII, 459). I had previously had no knowledge of any anticipation of this method of dealing with the theory of matrices except what is contained in Clifford's "Fragment on Matrices." Subsequently I examined Mr. Bucheim's proof, and I find that though it is sufficient to give the limits between which the nullity of the product of two matrices lies, nevertheless it is defective in assuming what is not always true, viz. that the resultant region of a matrix is coincident with the extension which in §9 I have denoted by B_{p+1}. Mr. Bucheim also omits to prove that the order of $\mathfrak{B}$ is the same as that of $\phi\mathfrak{B}$.

will appear presently, the latent region of ϕ corresponding to its latent root zero. *Hence if the null-region of ψ has no part in common with the latent region of ϕ corresponding to the latent root zero, the nullity of $\psi\phi$ is the sum of the nullities of ϕ and ψ.* *A fortiori*, if the latent regions of ϕ and ψ corresponding to the latent root zero are mutually exclusive, the nullity of $\psi\phi$ (and also of $\phi\psi$) is the sum of the nullities of ϕ and ψ.

If the order of the vacuous region of ϕ is zero, so that the null-region of ϕ is coextensive with the latent region of ϕ corresponding to the latent root zero, the null-region of $\psi\phi$ is the aggregate of A and C; hence if these extensions do not intersect, the nullity of $\psi\phi$ is $m+n$. Conversely, *if the order of the vacuous region of ϕ is zero, and the nullity of $\psi\phi$ is $m+n$, then the null-regions of ϕ and ψ have no part in common.*

Nullity of the Factors of the Identical Equation.

§11. If the latent roots of ϕ are all distinct, the nullity of the product $\Phi_1 = (\phi - g_1)(\phi - g_2) \ldots . (\phi - g_m)$ of m factors of the identical equation is m.* This theorem is Prof. Sylvester's, and is termed by him *the corollary of the law of nullity.* His demonstration is as follows: Since each factor, being simply vacuous, has a nullity of order unity, the nullity of the product Φ_1 cannot exceed m. Similarly the nullity of the product Φ_2 of the remaining factors of the identical equation cannot exceed $\omega - m$. But the nullity of the product of Φ_1 and Φ_2 is ω, and consequently the sum of their nullities must be as great as ω. Hence the nullity of Φ_1 cannot fall short of m, and the nullity of Φ_2 cannot fall short of $\omega - m$.

When the latent roots of ϕ are not all distinct, the law is not so simple. Suppose the distinct latent roots of ϕ be i in number, namely, $g_1, g_2, \ldots . g_i$, occurring severally $m_1, m_2, \ldots . m_i$ times, and that the identical equation is

$$(\phi - g_1)^{m_1 - \kappa_1}(\phi - g_2)^{m_2 - \kappa_2} \ldots . (\phi - g_i)^{m_i - \kappa_i} = 0 .$$

For the investigation of this case the following lemma is required, namely, that the vacuity of any positive integer power of $\phi - g$, for any latent root g, is equal to the vacuity of $\phi - g$, and hence to the number of times the latent root g occurs. This lemma is an immediate consequence of the theorem that the determinant of the product of two matrices is the product of their determinants;

* From the law of latency, §15, it follows that the vacuity of Φ_1 is also m.

for then, if m is a positive integer, if μ is a primitive m^{th} root of unity, and if $\psi = \phi - g$,

$$|\psi^m - g^m| \equiv |\psi - g| \cdot |\psi - \mu g| \cdot |\psi - \mu^2 g| \ldots . |\psi - \mu^{m-1} g| .$$

But if g^r is the lowest power of g that appears in the latent function $|\psi - g|$ of ψ, then $(\mu g)^r$ is the lowest power of μg that appears in $|\psi - \mu g|$, etc.; whence the lowest power of g^m that appears in $|\psi^m - g^m|$ is the r^{th} power of g^m. Consequently $\psi^m = (\phi - g_1)^m$ has the same vacuity as $\psi = \phi - g_1$.* Resuming the investigation of the nullity of any product of matrices $\phi - g$ for different latent root g, as a consequence of this lemma, the several factors $(\phi - g_1)^{m_1 - \kappa_1}$, $(\phi - g_2)^{m_2 - \kappa_2}$, etc., of the identical equation have the vacuities m_1, m_2, etc.; and hence their several nullities cannot exceed m_1, m_2, etc., respectively. But the nullity of their product, which cannot exceed the sum of their nullities, is ω. Hence the nullities of the several factors are m_1, m_2, etc., respectively; and hence corresponding to each latent root g of ϕ is an extension of order equal to the number of times that latent root is repeated which is annulled by that power of $\phi - g$ appearing in the identical equation.† These extensions I term the latent extensions or *latent regions* of the latent roots. In a similar way it may be shown that the nullity of the product of any two factors of the identical equation $(\phi - g_1)^{m_1 - \kappa_1}$ and $(\phi - g_2)^{m_2 - \kappa_2}$ is $m_1 + m_2$; but since the vacuity of either matrix is equal to its nullity, the order of the vacuous region of either matrix is zero; and hence by §10 their null-regions have no part in common. Similarly with respect to the null-regions of any two other factors of the identical equation. Hence the latent regions are mutually exclusive.‡ Since each positive integer power of $\phi - g_1$ is vacuous, there is, evidently, an extension of order at least unity annulled by $\phi - g_1$; an extension annulled by $(\phi - g_1)^2$, etc.; and each of these extensions is included in the one corresponding to the next higher power of $\phi - g_1$ in the series, since if an extension is annulled by any power of $\phi - g_1$ it is annulled by the next higher power. But as the nullity of no power of $\phi - g_1$ is greater than m_1, which is the order of the latent region corresponding to g_1, all these extensions are included in this latent region. Similarly with respect to the other latent roots. The latent region of

* It should be remembered that the latent roots of a matrix are the roots of its latent function.

† In this method of finding the nullity of the several factors $(\phi - g_1)^{m_1 - \kappa_1}$ of the identical equation, I have followed another one of Sylvester's methods of demonstrating the corollary of the law of nullity when the latent roots are all distinct, the only case Sylvester considers.

‡ That the latent extensions are mutually exclusive may be proved very simply by the method employed in the first part of §12.

ϕ corresponding to the latent roots g_1, g_2, etc., are, however, obviously the respective latent regions corresponding to the latent root zero of the matrices $\phi - g_1$, $\phi - g_2$, etc.; and consequently these extensions are also respectively the latent regions corresponding to the latent root zero of any positive integer powers of $\phi - g_1$, $\phi - g_2$, etc. Whence the nullity of the product whose factors are powers of $\phi - g_1$, $\phi - g_2$, etc., is by §10 the sum of the nullities of the several factors, as the latent regions of these factors corresponding to the latent root zero are mutually exclusive.

The next problem is to find the nullity of successive powers of ϕ less than any of its latent roots; and I shall show that the nullity of the $(\phi - g_1)^2$ is greater than the nullity of $\phi - g_1$ by an amount at least unity (unless the nullity of $\phi - g_1$ is m_1), and that the nullity of successive powers of $\phi - g_1$ goes on increasing by an amount not greater than the increment of nullity in the preceding power, until some power of $\phi - g_1$ is reached whose nullity is equal to its vacuity; that power of $\phi - g_1$ whose vacuity is equal to its nullity is evidently the factor in the identical equation corresponding to the latent root g_1. This theorem has already been proved in §9, where it was shown that the region annulled by $(\phi - g_1)^2$ consisted of A the null region of $\phi - g_1$, and of B_1 the extension transformed by $\phi - g_1$ into A, of order equal to or less than that of A, etc. The theorem may also be proved as follows: Denote $\phi - g_1$ by ψ, let the null region of ψ of order a be denoted by A; and, of the complementary extension, let B_1 of order b be that part transformed by ψ into A. Such an extension must exist, otherwise ψ^2 would annul only the extension A; hence A would also be the null region of ψ^3, since ψ^3 can annul only the null region of ψ^2 together with that extension projected into it by ψ, etc. Finally, no power of ψ would annul an extension other than A; but then the factor ψ would occur in the identical equation only to the first power; consequently the nullity of ψ would be as great as its vacuity. The order of B_1 cannot be greater than the order of A, for then the ψ of any b linearly independent quantities $\beta_1, \beta_2, \ldots . \beta_b$, in the region B_1 would be expressible linearly in terms of $a < b$ linearly independent quantities in the region A; hence for some value of the t's other than all zero, the expression

$$t_1\psi\beta_1 + t_2\psi\beta_2 + \ldots . + t_b\psi\beta_b = \psi(t_1\beta_1 + t_2\beta_2 + \ldots . + t_b\beta_b)$$

would vanish, which is contrary to supposition, since no part of B_1 is in the null region of ψ. In the same way it may be shown that if the nullity of ψ^{m+1} is not greater than the nullity of ψ^m, no higher power of ψ has a nullity greater than the nullity of ψ^m, which must therefore be equal to its vacuity; and also that the

extension annulled by ψ^{n+2} additional to that annulled by ψ^{n+1} cannot be of order greater than the order of the extension annulled by ψ^{n+1} additional to the null region of ψ^n.

§12. No rational integral function of ϕ operating upon any quantity in the latent region corresponding to one latent root can transform it wholly or in part into the latent region of any other latent root. By Sylvester's formula, §16, it may be shown that any function of ϕ may be reduced to a rational integral function; whence the above proposition may be stated for any function of ϕ. It is obviously only necessary to prove that no function of ϕ can transform any quantity in one latent region wholly into the latent region of another latent root. Suppose $F\phi$ is a rational integral function of ϕ of order n; let ξ_1 be a quantity in the region of g_1. If $F\phi.\xi_1 = \xi_2$, where ξ_2 is a quantity in the latent region of g_2, then for some integer $m \geqq m_2 - \varkappa_2$,

$$\begin{aligned}((\phi - g_1)^m + (g_1 - g_2)(\phi - g_1)^{m-1} + \ldots + (g_1 - g_2)^{m-1}(\phi - g_1) + (g_1 - g_2)^m)\, F\phi\xi_1 \\ = (\phi - g_2)^m \xi_2 = 0.\end{aligned}$$

Suppose $F\phi$ contains $\phi - g_1$ as a factor to the p^{th} power, then unless $F\phi\xi_1 = 0$, a case which need not be considered since ξ_2 is to be regarded as non-evanescent, there exists a linear relation

$$(\phi - g_1)^{m+n}\xi_1 + A(\phi - g_1)^{m+n-1}\xi_1 + \ldots + L(\phi - g_1)^{p+1}\xi_1 + M(\phi - g_1)^p\xi_1 = 0,$$

where $M \neq 0$, between $(\phi - g_1)^p\xi_1$ and other quantities that are annulled by successive powers of $\phi - g_1$, which is impossible.

Since the latent regions are mutually exclusive and together make up the extension of the ground, a number of linearly independent quantities may be taken from each latent region and together may be employed to represent the ground.

By what has just been proved, each latent region in respect to ϕ constitutes a subordinate ground; for the effect of the matrix ϕ upon any one of these extensions is to convert a set of linearly independent quantities, equal in number to its order, into other quantities in the same extension. It would thus seem that ϕ might be regarded as an aggregate of subordinate matrices corresponding to and equal in number to the distinct latent roots, each of which would have upon the latent region corresponding to it the effect of ϕ, and would have a null effect upon any other extension. This suggestion may be verified as follows: Let the latent roots of ϕ be i in number, namely, $g_1, g_2 \ldots g_i$, occurring severally $m, n, \ldots p$ times; let the latent regions corresponding to the g's be $A, B, \ldots L$

respectively; and, finally, let any m linearly independent quantities $\alpha_1, \alpha_2, \ldots \alpha_m$ be selected from A, any n linearly independent quantities $\beta_1, \beta_2, \ldots \beta_n$ be selected from B, etc., and let any p linearly independent quantities $\lambda_1, \lambda_2, \ldots \lambda_p$ be selected from L; these quantities together embrace the extension of the ground. If

$$\begin{aligned}
\phi\alpha_1 &= a_{11}\alpha_1 + a_{21}\alpha_2 + \ldots + a_{m1}\alpha_m, \\
\phi\alpha_2 &= a_{12}\alpha_1 + a_{22}\alpha_2 + \ldots + a_{m2}\alpha_m, \text{ etc.,} \\
\phi\beta_1 &= b_{11}\beta_1 + b_{21}\beta_2 + \ldots + b_{n1}\beta_n, \\
\phi\beta_2 &= b_{12}\beta_1 + b_{22}\beta_2 + \ldots + b_{n2}\beta_n, \text{ etc.,} \\
&\text{etc., etc.,} \\
\phi\lambda_1 &= l_{11}\lambda_1 + l_{21}\lambda_2 + \ldots + l_{p1}\lambda_p, \\
\phi\lambda_2 &= l_{12}\lambda_1 + l_{22}\lambda_2 + \ldots + l_{p2}\lambda_p, \text{ etc.,}
\end{aligned}$$

then the form of ϕ becomes

$$\left|\begin{array}{c|c|c|c}
\begin{matrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{matrix} & & & \\ \hline
 & \begin{matrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{matrix} & & \\ \hline
 & & \ddots & \\ \hline
 & & & \begin{matrix} l_{11} & l_{12} & \ldots & l_{1p} \\ l_{21} & l_{22} & \ldots & l_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ l_{p1} & l_{p2} & \ldots & l_{pp} \end{matrix}
\end{array}\right|$$

In this form all the constituents except those in the squares along the diagonal are zero. It is obvious from this form that the array of the a's forms a matrix by itself, and that its effect upon any region other than that of the latent root g_1 is to annul it, etc. Whence if ϕ_1 denote the matrix formed from the a's, and ϕ_2 that formed from the b's, etc., then ϕ_1, ϕ_2, etc., are nil-factorial together, and

$$\phi = \phi_1 + \phi_2 + \ldots\ldots + \phi_i,$$
$$\therefore\ \phi^2 = \phi_1^2 + \phi_2^2 + \ldots\ldots + \phi_i^2;$$

and, in general, $$F\phi = F\phi_1 + F\phi_2 + \ldots\ldots + F\phi_i.$$

The matrices ϕ_1, ϕ_2, etc., regarded as pertaining to the ground of ϕ, are, of course, vacuous; thus, from this point of view, ϕ_1 has the representation

$$\left(\begin{array}{cccccc} a_{11} & a_{12} & \ldots\ldots & a_{1m} & 0 \ldots\ldots & 0 \\ a_{21} & a_{22} & \ldots\ldots & a_{2m} & 0 \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mm} & 0 \ldots\ldots & 0 \\ 0 & 0 & \ldots\ldots & 0 & 0 \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots\ldots & 0 & 0 \ldots\ldots & 0 \end{array}\right)$$

and has a nullity at least $\omega - m$; if ϕ is non-vacuous, the nullity of ϕ_1 is just $\omega - m$, etc. But regarded as subordinate matrices of different systems, pertaining to the subordinate grounds $A, B, \ldots\ldots L$, then, unless ϕ is vacuous, $\phi_1, \phi_2, \ldots\ldots \phi_i$ must be considered as non-vacuous matrices; and ϕ_1 will have the representation

$$\left(\begin{array}{cccc} a_{11} & a_{12} & \ldots\ldots & a_{1m} \\ a_{21} & a_{22} & \ldots\ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mm} \end{array}\right)_A,$$

this matrix being supposed to operate only upon that part of the ground comprised in the subordinate ground A, namely, the extension of the set $(\alpha_1, \alpha_2, \ldots \alpha_m)$, and must be regarded as not susceptible of operating upon any other ground. The subscript A is employed to denote that the ultimate operands of this matrix are only expressions linear in the α's, etc. From this point of view it is proper to consider ϕ_1, ϕ_2, etc., as having reciprocals, providing ϕ is not vacuous; and then $$\phi^{-1} = \phi_1^{-1} + \phi_2^{-1} + \ldots\ldots + \phi_i^{-1}.$$

If ϕ is vacuous it must have zero as a latent root; but only one of the subordinate matrices corresponds to this root; consequently one and only one of the subordinate matrices will be vacuous, and evidently the vacuous subordinate matrix will have exactly the same vacuity and nullity as ϕ. The vacuous subordinate matrix is evidently a nilpotent quantity, since if ϕ is raised to a sufficiently higher power it annuls the region corresponding to the latent root zero. Thus if $g_1 = 0$, then A is annulled by $\phi^{m_1-\kappa_1}$; hence

$$\phi^{m_1-\kappa_1} = 0 + \phi_2^{m_1-\kappa_1} + \ldots . + \phi_i^{m_1-\kappa_1}.$$

The conception of a matrix as a sum of a set of subordinate matrices is more readily grasped by the consideration of Peirce's linear-form representation of a matrix. Thus

$$\phi = \Sigma_r \Sigma_s a_{rs} (\alpha_r : \alpha_s) + \Sigma_r \Sigma_s b_{rs} (\beta_r : \beta_s) + \ldots . + \Sigma_r \Sigma_s l_{rs} (\lambda_r : \lambda_s).$$

But evidently the set of vids $(\alpha_r : \alpha_s)$ form a quadrate system by themselves, similarly with respect to the set of vids $(\beta_r : \beta_s)$, etc., and any linear expressions in the vids $(\alpha_r : \alpha_s)$ will have as ultimate operands only expressions linear in the α's; while any expression linear in two different sets of vids, if they are to be regarded as susceptible of operating upon each other, are mutually nil-facient and nil-faciend.

Considering the vids $(\alpha_r : \alpha_s)$ as a quadrate system by themselves, scalar unity will be expressed by $\Sigma_1^m r\,(\alpha_r : \alpha_r)$, and may be denoted by 1_1. The quadrate system formed from the vids $(\beta_r : \beta_s)$ will have as its scalar unity $\Sigma_1^n r\,(\beta_r : \beta_r)$, which may be denoted by 1_2, etc. We evidently have

$$1_1 1_2 = 1_2 1_1 = 1_1 1_3 = 1_3 1_1 = \text{etc.} = 0;$$

and, denoting by 1 the unity of the complete system,

$$1 = 1_1 + 1_2 + \ldots . + 1_i.$$

Thus $$\phi - g = (\phi_1 - g 1_1) + (\phi_2 - g 1_2) + \ldots . + (\phi_i - g 1_i).$$

Since any matrix can thus be resolved into an aggregate of as many other matrices (mutual nil-factorial), as it has latent roots, each subordinate matrix corresponding to a latent root and being of order equal to the number of times that latent root occurs, hence in general it suffices to prove a theorem

relating to a single matrix for one all of whose latent roots are equal, when, if true, it may by means of this proposition be inferred for any case.*

§13. Having obtained the law governing the nullity of the factors of the identical equation, it is now possible to solve the problem touched upon in §3 with regard to a matrix subject to a condition involving only itself. As was stated above, it follows from Sylvester's formula that any condition equation to which a matrix is subject may be reduced to a rational integral equation. Let

$$F\phi \equiv (\phi - g_1)^{\alpha}(\phi - g_2)^{\beta} \ldots\ldots (\phi - g_m)^{\sigma} \ldots\ldots (\phi - g_n)^{\tau} = 0$$

be the rational integral equation expressing the condition to which ϕ is subject; then ϕ will satisfy this condition if any $m \leqq \omega$ of the g's are selected as its latent roots, as $g_1, g_2, \ldots\ldots g_m$, each occurring $\alpha', \beta', \ldots\ldots \sigma'$ times respectively; and if $(\phi - g_1)^{\alpha' - \kappa_1}$ has the nullity α', $(\phi - g_2)^{\beta' - \kappa_2}$ the nullity β', etc., and $(\phi - g_m)^{\sigma' - \kappa_m}$ has the nullity σ', provided the sum of the accented Greek letters is ω, and $\alpha' - \kappa_1 \leqq \alpha$, $\beta' - \kappa_2 \leqq \beta$, etc. For then

$$(\phi - g_1)^{\alpha' - \kappa_1}(\phi - g_2)^{\beta' - \kappa_2} \ldots\ldots (\phi - g_m)^{\sigma' - \kappa_m} = 0,$$
$$\therefore\ (\phi - g_1)^{\alpha}(\phi - g_2)^{\beta} \ldots\ldots (\phi - g_m)^{\sigma} \ldots\ldots (\phi - g_n)^{\tau} = 0.$$

Conversely, if ϕ satisfies the last equation it must have a certain number of the g's as latent roots, each occurring a sufficient number of times to make the total number of latent roots equal to ω; and if g is any one of these latent roots, some power of $\phi - g$ equal to or lower than that which appears in the above equation must have a nullity equal to the number of times g occurs.

Law of Congruity and Law of Latency.

§14. If ρ_1 is an axis of ϕ, it is an axis of any integer power ϕ^m of ϕ; for if $\phi\rho_1 = g_1\rho_1$, hence $\phi^m\rho_1 = g_1^m\rho_1$. Similarly with respect to the other axes of ϕ. Hence the axes of ϕ are all axes of ϕ^m; but since the axes of ϕ may be linearly related, they do not necessarily constitute all the axes of ϕ^m. In like manner the axes of any root $\phi^{\frac{1}{n}}$ (where n is an integer) are comprised among the axes of ϕ. We have for each latent root g_r of ϕ, v being a primitive n^{th} root of unity, the equation

* Thus if the identical equation has been proved for a matrix of any order whose latent roots are all equal, it may be shown by the principle of this section to hold in any case. For if

$$\phi = \phi_1 + \phi_2 + \ldots\ldots + \phi_i,$$

and

$$0 = (\phi_1 - g_1 1_1)^{m_1} = (\phi_2 - g_2 1_2)^{m_2} = \text{etc.},$$
$$\therefore\ (\phi - g_1)^{m_1}(\phi - g_2)^{m_2} \ldots\ldots (\phi - g_i)^{m_i} = 0.$$

$$(\phi^{\frac{1}{n}} - g_r^{\frac{1}{n}})(\phi^{\frac{1}{n}} - vg_r^{\frac{1}{n}})(\phi^{\frac{1}{n}} - v^2 g_r^{\frac{1}{n}}) \ldots (\phi^{\frac{1}{n}} - v^{n-1}g_r)\,\rho_r = (\phi - g_r)\,\rho_r = 0.$$

Evidently one at least of the matrices which in the first member operate upon ρ_r must be vacuous, and hence one of the n^{th} roots of each distinct latent roots of ϕ is a latent root of $\phi^{\frac{1}{n}}$. If then the latent roots g_1, g_2, etc., of ϕ are all distinct, so also are the latent roots of $\phi^{\frac{1}{n}}$; and then, by §2, the axes of ϕ, ρ_1, ρ_2, etc., are ω linearly independent and determinate quantities, as are also the axes of $\phi^{\frac{1}{n}}$; and since the latter are comprised among the axes of ϕ, it is evident that every axis of ϕ is an axis of $\phi^{\frac{1}{n}}$. Hence every axis of ϕ is an axis of $\phi^{\frac{m}{n}}$; and for the ω axes we have the equations $\phi^{\frac{m}{n}}\rho_1 = g_1^{\frac{m}{n}}\rho_1$, $\phi^{\frac{m}{n}}\rho_2 = g_2^{\frac{m}{n}}\rho_2$, etc. It may now be established by the method of limits, in conformity with the definition of I, §8, that for any scalar m we have for the ω axes the equations $\phi^m\rho_1 = g_1^m\rho_1$, $\phi^m\rho_2 = g_2^m\rho_2$, etc. Consequently $\Sigma k\phi^m.\rho_1 = \Sigma k g_1^m.\rho_1$, $\Sigma k\phi^m\rho_2 = \Sigma k g_2^m.\rho_2$, etc., provided the coefficients and exponents are scalars; and so if $F\phi$ is any function of ϕ formed by the addition of scalar multiples of scalar powers of ϕ, then for the ω axes $F\phi\rho_1 = Fg_1.\rho_1$, $F\phi\rho_2 = Fg_2.\rho_2$, etc. Thus what has been proved is, *when the latent roots of ϕ are all distinct, every axis of ϕ is an axis of $F\phi$*; and since the above equations, or their equivalent, $(F\phi - Fg_1)\,\rho_1 = 0$, $(F\phi - Fg_2)\,\rho_2 = 0$, etc., hold only for the latent roots of $F\phi$, hence *when the latent roots of ϕ are all distinct, the latent roots of $F\phi$ are the same function of the latent roots of ϕ.*

The first of these theorems I term the *law of congruity of the axes.* The second is very important; it is due to Prof. Sylvester, who terms it the *law of latency.* In the next section I shall extend the proof of the law of latency to the general case. The term law of congruity was employed by Prof. Sylvester interchangeably with the term law of latency.

The converse of the law of congruity is also true, subject to a slight modification when two or more of the latent roots of $F\phi$ become equal. In this case, if the axes of $F\phi$ are first selected, they may not all prove to be axes of ϕ; but when this occurs, other quantities may be chosen which, together with the axes common to both ϕ and $F\phi$, shall constitute the ω axes of $F\phi$ and also be axes of ϕ. Thus, to take an example, let ρ_1, ρ_2, ρ_3 be the three axes of the ternary

matrix ϕ whose corresponding latent roots are $1, -1, 2$ respectively; then the axis of ϕ^2 corresponding to the latent root unity is indefinite; for

$$\phi^2(x\rho_1 + y\rho_2) = x\rho_1 + y\rho_2.$$

In general, this quantity will not be an axis of ϕ. However, ρ_1, ρ_2 and ρ_3 are axes of both ϕ and ϕ^2.

If the latent roots of ϕ are all distinct, the ground may be represented in terms of the axes of ϕ; in which case, if $F\phi$ is any function of ϕ involving only the matrix ϕ and unity, we have obviously

$$F\phi = \begin{vmatrix} Fg_1 & 0 & \dots & 0 \\ 0 & Fg_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & & Fg_\omega \end{vmatrix}$$

From this form of $F\phi$ immediate proofs for the general case of the law of latency and of the law of congruity may be derived.

§15. If the latent roots of ϕ are all distinct, it is not necessarily true that the latent roots of $F\phi$ are all distinct. When this is not the case, the order of the identical equation in $F\phi$ is lowered by unity for each latent root of $F\phi$ which becomes equal to another. Thus suppose $Fg_1 = Fg_2 = \dots = Fg_n$, then $\rho_1, \rho_2, \dots \rho_n$ are all annulled by $F\phi - Fg_1$; consequently the application of this operator to the most general expression ρ in the ground (which can now, by §2, be expressed in terms of the ω axes of ϕ) annuls n components and leaves $\omega - n$ components which will be annulled by the remaining factors of the identical equation in $F\phi$. I. e. the degree of the identical equation is lowered by $n-1$.

The law of latency may be proved in the general case as follows: Let the latent roots of ϕ be all distinct, and let $g_2 - g_1 = h_2$, etc.; as h diminishes, $Fg_2 = Fg_1 + h_2F'g_1 +$ etc. approaches Fg_1. In the limit ϕ has two equal roots g_1, and $F\phi$ has two latent roots equal to Fg_1. Obviously, when ϕ has two or more equal latent roots, the latent roots of $F\phi$ corresponding to these are not necessarily equal, since $F\phi$ may be a many-valued function. Thus, if $(\phi - g_1)^2 = 0$, the latent roots of $\phi^{\frac{1}{2}}$ are $\pm\sqrt{g_1}$, and the identical equation in $\phi^{\frac{1}{2}}$ is

$$(\phi^{\frac{1}{2}} - \sqrt{g_1})(\phi^{\frac{1}{2}} + \sqrt{g_1}) = 0.$$

Sylvester's Formula.

§16. Sylvester has given without demonstration the following theorem in the Johns Hopkins Univ. Circ. (No. 28, 1884):

$$F\phi = \Sigma Fg_1 . \frac{(\phi - g_2)(\phi - g_3) \ldots . (\phi - g_\omega)}{(g_1 - g_2)(g_1 - g_3) \ldots . (g_1 - g_\omega)}.$$

It may be proved as follows: Suppose the latent roots of ϕ are all distinct, and denote for convenience the left-hand member of the above equation by (Σ); then for the ω axes of ϕ we have $(\Sigma)\,\rho_1 = Fg_1.\rho_1$, $(\Sigma)\,\rho_2 = Fg_2.\rho_2$, etc. But by the law of congruity and the law of latency, $F(\phi)\,\rho_1 = Fg_1.\rho_1$, $F(\phi)\,\rho_2 = Fg_2.\rho_2$, etc., for the ω axes; consequently $(\Sigma)\,\rho_1 = F(\phi)\,\rho_1$, $(\Sigma)\,\rho_2 = F(\phi)\,\rho_2$, etc.; and hence for any quantity linear in the axes $\rho = z_1\rho_1 + z_2\rho_2 +$ etc., $F(\phi)\,\rho = (\Sigma)\,\rho$. Since the latent roots of ϕ are all supposed distinct, ρ may be any quantity whatever in the ground of which ϕ is a linear unit function. Hence

$$F\phi = (\Sigma).$$

As this mode of proof is a verification, if $F\phi$ is a many-valued function, it is necessary to show that Sylvester's formula gives all possible solutions of $\psi = F(\phi)$. Let the latent roots of ϕ be all distinct; take as latent roots of a matrix ψ any set of the values of Fg_1, Fg_2, etc., and as axes corresponding respectively to them the ω linearly independent axes of ϕ, ρ_1, ρ_2, etc.; then ψ satisfies the equation $\psi.\rho = F\phi.\rho$ for each of these axes, and consequently for any quantity in the ground. Evidently m^ω matrices ψ may thus be formed if m is the number of values of the function Fg, and only the matrices so formed satisfy for all values of ρ the condition $\psi\rho = F\phi.\rho$, and thus are solutions of $\psi = F\phi$. These m^ω matrices are, however, all contained in Sylvester's formula.

By the theory of limits the theorem may be extended to the case where the latent roots of ϕ are not all distinct.

In the general case in which the latent roots of ϕ are all distinct, a simple proof of Sylvester's formula may also be derived from the form of $F\phi$ when the axes of ϕ represent the ground. In §5 it was shown that when the axes of ϕ are linearly independent, whatever the set of quantities chosen to represent the ground, there is always a definite matrix ϖ such that

$$\phi = (\varpi)\left|\begin{matrix} g_1 & 0 & \ldots . & 0 \\ 0 & g_2 & \ldots . & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots . & g \end{matrix}\right|(\varpi^{-1}),$$

whence the form of ϕ being given in terms of any set of quantities, we can very simply find the form of $F\phi$ in terms of the same set of quantities, for

$$F\phi = (\varpi)\begin{vmatrix} Fg_1 & 0 & \dots & 0 \\ 0 & Fg_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Fg_\omega \end{vmatrix}(\varpi)^{-1}.$$

If ϕ is a scalar, any quantity in the ground is an axis, hence the above considerations showing that Sylvester's formula gives all the solution of $\psi = F\phi$ do not apply. And on inspection it is evident that the formula does not give the non-scalar roots of a scalar. These must therefore be formed by an independent investigation. There are evidently two cases to be considered, the roots of the matrix zero and the roots of the matrix unity; from the latter the roots of any non-zero matrix may be found.

Roots of the Matrix Zero.

§17. It is very evident by §13 that the latent roots of a nilpotent quantity are all zero, and, conversely, that a matrix all of whose latent roots are zero is nilpotent. Whence it follows that the number of roots of zero with any index is infinite, meaning by index of a root of a matrix the least power of the root which will reproduce the matrix. Of the square roots of zero in a matrix of order ω there are $\omega^2 - 1$ linearly independent. Thus, if $\omega = 2$, the four vids of a dual matrix may be expressed linearly in terms of

$$(A:A) + (B:B),\ (A:A) - (B:B) + (A:B) - (B:A),\ (A:B) \text{ and } (B:A),$$

of which the first is unity and the other three square roots of zero. The proposition may be shown similarly for any value of ω.

No root of zero can have an index greater than ω. For, as Benjamin Peirce has shown, there is no linear relation possible between the powers of a nilpotent quantity that does not vanish; but if ϕ is a root of zero, unless its ω^{th} or some lower power vanishes, by means of the identical equation, the ω^{th} power of ϕ can be expressed linearly in terms of the lower powers, the coefficients not all being zero.

If ϕ is an m^{th} root of zero, then by §11 the nullity of successive powers of ϕ increases until the m^{th} power is reached; and the nullity of ϕ^2 is at most twice

the nullity of ϕ, the nullity of ϕ^3 exceeds the nullity of ϕ^2 by an amount not greater than the increment of the nullity of ϕ^2 over the nullity of ϕ, etc.* Following the reasoning of §9, let the extension of the p linearly independent quantities $\alpha_1, \alpha_2, \ldots . \alpha_p$ be the null region of ϕ; let the extension of the q linearly independent quantities $\beta_1, \beta_2, \ldots . \beta_q$ be that transformed by ϕ into the null region of ϕ; let the extension of the r linearly independent quantities $\gamma_1, \gamma_2, \ldots . \gamma_r$ be that transformed by ϕ into the extension of the β's, etc.; finally let the s linearly independent quantities $\varkappa_1, \varkappa_2, \ldots . \varkappa_s$ be that transformed by ϕ into the extension annulled by ϕ^{m-2} additional to that annulled by lower powers of ϕ; and let the extension of the t linearly independent quantities $\lambda_1, \lambda_2, \ldots . \lambda_t$ constitute the remaining extension of the ground. Then these sets of quantities, the α's, β's, etc., together embrace the extension of the ground, and

$$\phi\alpha_1 = \phi\alpha_2 = \ldots . = \phi\alpha_r = 0,$$

$$\begin{aligned} \phi\beta_1 &= a_{11}\alpha_1 + a_{21}\alpha_2 + \ldots . + a_{p1}\alpha_p, \\ \phi\beta_2 &= a_{12}\alpha_1 + a_{22}\alpha_2 + \ldots . + a_{p2}\alpha_p, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi\beta_q &= a_{1q}\alpha_1 + a_{2q}\alpha_2 + \ldots . + a_{pq}\alpha_p, \end{aligned}$$

$$\begin{aligned} \phi\gamma_1 &= b_{11}\beta_1 + b_{21}\beta_2 + \ldots . + b_{q1}\beta_q, \\ \phi\gamma_2 &= b_{12}\beta_1 + b_{22}\beta_2 + \ldots . + b_{q2}\beta_q, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi\gamma_r &= b_{1r}\beta_1 + b_{2r}\beta_r + \ldots . + b_{qr}\beta_q, \end{aligned}$$

etc.; finally,

$$\begin{aligned} \phi\lambda_1 &= k_{11}\varkappa_1 + k_{21}\varkappa_2 + \ldots . + k_{s1}\varkappa_s, \\ \phi\lambda_2 &= k_{12}\varkappa_1 + k_{22}\varkappa_2 + \ldots . + k_{s2}\varkappa_s, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi\lambda_t &= k_{1t}\varkappa_1 + k_{2t}\varkappa_2 + \ldots . + k_{st}\varkappa_s. \end{aligned}$$

---

* Hence certain roots of nilpotent quantities have no representation. Thus let ϕ be a matrix of order six whose nullity is two, and whose fourth power and no lower power vanishes; then ϕ has no square root. For if $\psi^2 = \phi$, then, since ψ has at least simple nullity, the nullity of $\psi^6 = \phi^3$ is at least six: hence $\phi^3 = 0$.

Whence the form now representing ϕ is

$$\left|\begin{array}{c|c|c|c|c}
 & \begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2q} \\ \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots & a_{pq} \end{array} & & & \\
\hline
 & & \begin{array}{cccc} b_{11} & b_{12} & \dots & b_{1r} \\ b_{21} & b_{22} & \dots & b_{2r} \\ \dots & \dots & \dots & \dots \\ b_{q1} & b_{q2} & \dots & b_{qr} \end{array} & & \\
 & & & \ddots & \\
 & & & & \begin{array}{cccc} k_{11} & k_{12} & \dots & k_{1t} \\ k_{21} & k_{22} & \dots & k_{2t} \\ \dots & \dots & \dots & \dots \\ k_{s1} & k_{s2} & \dots & k_{st} \end{array} \\
\hline
 & & & &
\end{array}\right|$$

In this form all the constituents except those in the rectangles bordered on the left and below by heavy lines are zero. The successive heavy lines at every other alternate corner intersect with the diagonal, and no one of them is longer than the next preceding one. The square at the left-hand upper corner and the square at the lower right-hand corner consist respectively of p^2 and t^2 zeros.

If ϕ is a root of zero, so also is its converse.

Roots of the Matrix Unity.

§18. If ϕ is an m^{th} root of unity it is subject to the condition

$$\phi^m - 1 \equiv (\phi - \lambda_1)(\phi - \lambda_2) \ldots (\phi - \lambda_m) = 0,$$

the λ's being the scalar m^{th} roots of unity. By §13, if any $n \leqq \omega$ of the m λ's, as $\lambda_1, \lambda_2, \ldots \lambda_n$ repeated respectively $k_1, k_2, \ldots k_n$ times (the sum of the k's being ω), are chosen as the latent roots of ϕ; and if $\phi - \lambda_1, \phi - \lambda_2, \ldots \phi - \lambda_n$ have the respective nullities $k_1, k_2, \ldots k_n$, then ϕ satisfies the condition; and conversely. It is obvious that the number of the non-scalar roots of unity with any index is infinite.* If no lower power of ϕ than the m^{th} is a scalar, then ϕ will be termed a *primitive* m^{th} root of unity. The condition necessary and sufficient that ϕ, in addition to being an m^{th} root of unity, shall be a primitive m^{th} root, is that one at least of its latent roots shall be primitive.

When the index $m \leqq \omega$, and the entire set of the scalar m^{th} roots of unity are latent roots of ϕ, then any m successive powers of ϕ are linearly independent. For then the identical equation is $\phi^m - 1 = 0$; but if there were any linear relation between m successive powers of ϕ, the identical equation would be of order $m - 1$, which, by §2, is impossible if ϕ has m distinct latent roots.

In the calculus of matrices, as in ordinary algebra, the m^{th} roots of any matrix may in general be obtained by taking any one of its m^{th} roots and multiplying it successively by all the m^{ω} m^{th} roots of unity of a certain set, namely, those that have as axes the axes of the matrix.

It is obvious that if ϕ is an m^{th} root of unity, so also is its converse, and if ϕ is primitive, so also is $\breve{\phi}$.

§19. *Every matrix whose order is a prime number has ω^2 linearly independent, primitive, ω^{th} roots of unity.* For, ω being prime, let ϕ denote the matrix

*If ϕ is not a scalar and has all its latent roots distinct, as was stated in §16, $\phi^{\frac{1}{m}}$ (where m is an integer) has in general m^{ω} values, obtained by combining each axis of ϕ with one of the m^{th} roots of the latent root of ϕ corresponding to that axis. The non-scalar roots of a scalar may similarly be obtained, but since any quantity is the axis of a scalar, these roots are infinite in number.

$$
\begin{array}{r|cccccccc|}
 & (1) & (2) & (\omega-\kappa) & (\omega-\kappa+1) & (\omega-\kappa+2) & (\omega-\kappa+3) & (\omega-1) & (\omega) \\
\hline
(1) & 0 & 0 & \ldots\ldots 0 & 0 & a_1 & 0 & \ldots\ldots 0 & 0 \\
(2) & 0 & 0 & \ldots\ldots 0 & 0 & 0 & a_2 & \ldots\ldots 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(\kappa-2) & 0 & 0 & \ldots\ldots 0 & 0 & 0 & 0 & \ldots\ldots a_{\kappa-2} & 0 \\
(\kappa-1) & 0 & 0 & \ldots\ldots 0 & 0 & 0 & 0 & \ldots\ldots 0 & a_{\kappa-1} \\
\kappa & a_\kappa & 0 & \ldots\ldots 0 & 0 & 0 & 0 & \ldots\ldots 0 & 0 \\
(\kappa+1) & 0 & a_{\kappa+1} & \ldots\ldots 0 & 0 & 0 & 0 & \ldots\ldots 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(\omega-1) & 0 & 0 & \ldots\ldots a_{\omega-1} & 0 & 0 & 0 & \ldots\ldots 0 & 0 \\
(\omega) & 0 & 0 & \ldots\ldots 0 & a_\omega & 0 & 0 & \ldots\ldots 0 & 0 \\
\hline
\end{array}
$$

in which all the constituents are zero except one in each column, and the non-zero constituents, forming a broken diagonal, are denoted each by a with subscript indicating the row in which it appears; the constituent in the κ^{th} row appears in the first column, that in the $(\kappa+1)^{\text{th}}$ row in the second column, and, in general, if $[x]$ denotes the smallest positive residue (modulus ω) of x, then the constituent in the $[\kappa+r-1]^{\text{th}}$ rows appears in the r^{th} column. The constituent in the first column I shall term the leading constituent. It is evident, then, if $(\alpha_1, \alpha_2, \ldots\ldots \alpha_\omega)$ are the elements of the set upon which ϕ operates,

$$\phi\alpha_1 = a_\kappa \alpha_\kappa, \quad \phi\alpha_2 = a_{\kappa+1}\alpha_{\kappa+1}, \quad \text{etc.},$$

and that the general expression for the ϕ of any one α_r of the α's is

$$\phi\alpha_r = a_{[\kappa+r-1]}\alpha_{[\kappa+r-1]}.$$

I. e. ϕ applied to the α's advances each by $\kappa-1$ places and multiplies it by a certain one of the a's. The application of ϕ^2 to each of the α's advances it by $2(\kappa-1)$ places, and multiplies it by the product of two of the a's, etc.; finally, ϕ^ω advances each α by a multiple of ω places, i. e., transforms each α into itself and multiplies it by the product of all the a's. If, now, λ be an imaginary ω^{th} root of unity, and the successive a's, $a_1, a_2, \ldots\ldots a_\omega$ are severally put equal to $1, \lambda, \lambda^2, \ldots\ldots \lambda^{\omega-1}$, denoting by ϕ_1 what ϕ then becomes, we have

$$\phi_1^\omega(\alpha_1, \alpha_2, \ldots\ldots \alpha_\omega) = \lambda^{\frac{\omega(\omega+1)}{2}}(\alpha_1, \alpha_2, \ldots\ldots \alpha_\omega).$$

Whence unless $\omega = 2$, ϕ_1 is an ω^{th} root of unity; and since ω is prime, it is readily seen that no lower power of ϕ_1 is a scalar. Similarly, denoting by ϕ_2 what ϕ becomes

when the a's in the same order are put equal to $1, \lambda^2, \lambda^4, \ldots \lambda^{2(\omega-1)}$, then ϕ_2 is also a primitive ω^{th} root of unity; finally, denoting by $\phi_{\omega-1}$ what ϕ becomes when for the a's in the same order are substituted the $(\omega-1)^{\text{th}}$ powers of $1, \lambda, \lambda^2, \ldots \lambda^{\omega-1}$, then $\phi^{\omega}_{\omega-1}=1$ and also no lower power of $\phi_{\omega-1}$ is a scalar. In this way $\omega-1$ matrices are obtained which are primitive ω^{th} roots of unity; besides these, another primitive ω^{th} root may be obtained by putting the a's all equal to unity, and this matrix may be denoted by ϕ_0. These ω matrices are linearly independent, for otherwise, for some set of values of the x's, other than all equal to zero, we should have

$$\begin{aligned}
&x_0\phi_0 + x_1\phi_1 + x_2\phi_2 + \ldots + x_{\omega-1}\phi_{\omega-1} = 0,\\
\therefore\ &x_0 + x_1 + x_2 + \ldots + x_{\omega-1} = 0,\\
&x_0 + \lambda x_1 + \lambda^2 x_2 + \ldots + \lambda^{\omega-1}x_{\omega-1} = 0,\\
&x_0 + \lambda^2 x_1 + \lambda^4 x_2 + \ldots + \lambda^{2(\omega-1)}x_{\omega-1} = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&x_0 + \lambda^{\omega-1}x_1 + \lambda^{2(\omega-1)}x_2 + \ldots + \lambda^{(\omega-1)^2}x_{\omega-1} = 0,
\end{aligned}$$

which is impossible. Thus any position of the leading constituent other than in the first row (i. e. for any value of x other than $x=1$), a proper choice of the a's gives ω linearly independent ω^{th} roots of unity; and thus in this way $\omega(\omega-1)$ linearly independent ω^{th} roots of unity may be obtained; for it is evident that the matrices obtained for different values of x are linearly independent, being expressions in different vids. In addition to these, let $\omega-1$ matrices be formed by taking as constituents of the principal diagonal in the first matrix the set of scalar $1, \lambda, \lambda^2, \ldots \lambda^{\omega-1}$; in the second, the squares of this set, etc., and finally, the $(\omega-1)^{\text{th}}$ powers of this set for the $(\omega-1)^{\text{th}}$ matrix, all the other constituents in each matrix being zero. These $\omega-1$ matrices are primitive ω^{th} roots of unity, as may easily be seen by considering their effect upon the set $(\alpha_1, \alpha_2, \ldots \alpha_\omega)$. That they are linearly independent of each other and unity may be shown by the above reasoning; and since they are formed from the vids along the diagonal, which do not occur in the other set of $\omega(\omega-1)$ roots of unity, they can have no linear relation with these. Thus, in addition to unity (or rather to a primitive scalar ω^{th} root of unity), ω^2-1 other primitive ω^{th} root of unity may be formed, all of which are linearly independent.

If $\omega=2$, the four matrices formed in this way are

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \qquad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}.$$

Of these the first is unity, the second and third are square roots of unity, and the fourth is a square root of -1. By multiplying the fourth root by $\sqrt{-1}$, it is of course converted into a square root of unity. Thus a dual matrix may also be expressed linearly in terms of four square roots of unity.

Algebras Analogous to Quaternions.

§20. It was stated in I, §10, that if i, j, k are any three mutually normal unit vectors, any quaternion may be expressed linearly in terms of the four new units
$$\frac{1+i\sqrt{-1}}{2},\quad \frac{j+k\sqrt{-1}}{2},\quad \frac{-j+k\sqrt{-1}}{2},\quad \frac{1-i\sqrt{-1}}{2},$$
which, having the same multiplication table as the four vids of a dual matrix,
$$(A:A),\quad (A:B),\quad (B:A),\quad (B:B),$$
may, consequently, be regarded as respectively identical with them. This identification gives the following values for the ordinary quaternion units,
$$1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad i=\begin{pmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix},\quad j=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad k=\begin{pmatrix}0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix}.$$
The discovery of this form of quaternions, which I have termed the canonical form of quaternions (I, §8), as has been stated, is due to Benjamin Peirce; it received its full significance only after the discovery by his son, Charles Peirce, of the unlimited system of quadrates formed from the system of vids $(A:A)$, $(A:B)$, etc., when it appeared that quaternions was only the first of this system of quadrate algebras, and the identification of quaternions with the theory of dual matrices was virtually accomplished. Evidently of all linear associative algebras the quadrate algebras form a class which are closely related, and consequently are closely analogous to quaternions. In the preceding section it was shown that all matrices or quadrates whose order is a prime number may be regarded as linear in unity and ω^2-1 linearly independent, primitive ω^{th} roots of unity, just as quaternions is an algebra linear in unity and three square roots of -1, or of unity; whence the analogy between quaternions and these other quadrates extends beyond the quadrate form possessed by each. Moreover, these ω^2-1 roots of unity are formed from the ω^2 vids of the quadrate of order ω, except for a scalar factor, precisely as the i, j, k of quaternions are formed

from the vids of the dual matrix. Thus the quadrate algebra of prime order next in order to quaternions (nonions) is that formed from the vids of a triple matrix: comparing its eight cube roots of unity with the i, j, k of quaternions, we have

$$i = [(A:A) - (B:B)]\sqrt{-1}, \quad j = (A:B) - (B:A), \quad k = [(A:B) + (B:A)]\sqrt{-1},$$

while the nine nonion cube roots are

$$\begin{array}{ll} (A:A) + \lambda(B:B) + \lambda^2(C:C), & (A:A) + \lambda^2(B:B) + \lambda(C:C), \\ (A:B) + (B:C) + (C:A), & (A:B) + \lambda(B:C) + \lambda^2(C:A), \\ (A:B) + \lambda^2(B:C) + \lambda(C:A), & (A:C) + (B:A) + (C:B), \\ (A:C) + \lambda(B:A) + \lambda^2(C:B), & (A:C) + \lambda^2(B:A) + \lambda(C:B);^* \end{array}$$

and the representation of unity in the first system is $(A:A) + (B:B)$, and in the second $(A:A) + (B:B) + (C:C)$. It thus appears that there are an infinity of algebras exactly analogous to quaternions, namely, those formed from the vids of the matrices whose order is prime. I shall now proceed to show that this analogy may be still farther extended, and that all the algebras analogous to quaternions, and indeed matrices of any order, admit of selective symbols like the S and V of quaternions, are resolvable into the product of a versor and a tensor, and that there are functions similar to the conjugate of a quaternion, equal in number to $\omega - 1$, if ω is the order of the algebra. I shall show this in the case of nonions, but the proof will be applicable to any case.

§21. A quaternion q may, in general, be represented by

$$q = Sq + Vq = a + bi,$$

where i is a unit vector. In matrix form this is

$$q = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} = \begin{pmatrix} a + b\sqrt{-1} & 0 \\ 0 & a - b\sqrt{-1} \end{pmatrix}.$$

Hence $a + b\sqrt{-1} = Sq + TVq.\sqrt{-1}$, and $a - b\sqrt{-1} = Sq - TVq.\sqrt{-1}$ are the latent roots of q.

$$Kq = Sq - Vq = \begin{pmatrix} a - b\sqrt{-1} & 0 \\ 0 & a + b\sqrt{-1} \end{pmatrix},$$

* These units are the converse of those given by the method of the preceding section. The first of the series of quadrate algebras analogous to quaternions, nonions, was discovered independently by the Peirces and Sylvester. I have in I, §10, given a short account of this discovery.

i. e. the conjugate of q is obtained from q by interchanging its latent roots but leaving its axes unchanged.

Now a nonion, or matrix of the third order, provided its axes are taken to represent the ground, may in general be put in the form

$$\begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} = \begin{pmatrix} a+b+c & 0 & 0 \\ 0 & a+\lambda b+\lambda^2 c & 0 \\ 0 & 0 & a+\lambda^2 b+\lambda c \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} + c\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is an imaginary cube root of unity. The first of the matrices in the third member is unity, the second is a non-scalar cube root of unity, the third, being the square of the second, is also a non-scalar cube root of unity. Denoting the second matrix by i, n becomes $a+bi+ci^2$. To select the scalar and non-scalar parts of n, the selective symbols S and V may be employed; and to discriminate between the first and second parts of the non-scalar portion of n the V may be written with subscripts 1 and 2. Employing this notation, we have

$$n = Sn + V_1 n + V_2 n = a + bi + ci^2.$$

And since $V_1 n$ and $V_2 n$ are scalar multiples of non-scalar cube roots of unity, we have $V(V_1 n)^3 = V(V_2 n)^3 = 0$, just as in quaternions $V(Vq)^2 = 0$; and as in quaternions the last formula gives the important result $V(Vq.Vq' + Vq'.Vq) = 0$, so letting $V_1 n = \alpha_1$, $V_1 n' = \beta_1$, $V_1 n'' = \gamma_1$, and $V_2 n = \alpha_2$, $V_2 n' = \beta_2$, $V_2 n'' = \gamma_2$, from the two nonion formulae we may obtain in a similar way the two following results:

$$V(\alpha_1\beta_1\gamma_1 + \alpha_1\gamma_1\beta_1 + \beta_1\alpha_1\gamma_1 + \beta_1\gamma_1\alpha_1 + \gamma_1\alpha_1\beta_1 + \gamma_1\beta_1\alpha_1) = 0,$$

and a similar formula in which the α_1, β_1, γ_1 are replaced by α_2, β_2, γ_2.

As in quaternions $V^2 q = -T^2 Vq$, so in the algebra of nonions we may write

$$b^3 = V_1^3 n = T^3 V_1 n, \quad c^3 = V_2^3 n = T^3 V_2 n.$$

Since the conjugate of q is obtained by interchanging its latent roots, this suggests that a cyclic interchange of the latent roots of n, leaving its axes

unchanged, should produce a function of n similar to the conjugate of q; and the conjugate of n may be defined by the equation

$$Kn = \begin{pmatrix} g_2 & & \\ & g_3 & \\ & & g_1 \end{pmatrix} = Sn + \lambda V_1 n + \lambda^2 V_2 n .$$

The conjugate of Kn is KKn, which may be written $K^2 n$; and evidently

$$K^2 n = \begin{pmatrix} g_3 & & \\ & g_1 & \\ & & g_2 \end{pmatrix} = Sn + \lambda^2 V_1 n + \lambda V_2 n .$$

These formulae resemble that for the conjugate of q. In quaternions $K^2 = 1$, but in nonions we may write $K^2 = K^{-1}$, and $K^3 = 1$.

The tensor of a quaternion may be defined as the square root of the product of the quaternion and its conjugate. Following the analogy thus suggested, the tensor of a nonion n may be defined by the equation

$$T^3 n = n . Kn . K^2 n = a^3 + b^3 + c^3 - 3abc,$$
$$\therefore\ T^3 n = S^3 n + T^3 V_1 n + T^3 V_2 n - 3Sn . TV_1 n . TV_2 n,$$

which is the analogue of $T^2 q = S^2 q + T^2 Vq$. It is readily seen that the square of the tensor of a quaternion is equal to the product of its latent roots, and thus to its content; and similarly, that the cube of the tensor of a nonion is equal to the product of its latent roots, and hence to its content. Whence, since the determinant or content of the product of two matrices is equal to the product of their contents, the tensor of the product of two nonions is equal to the product of their tensors. When a nonion is expressed in terms of unity and eight non-scalar cube roots of unity, this proposition gives a theorem analogous to Euler's theorem when that is regarded as a theorem relating to the product of two quaternions; but the tensor of a nonion is then too complicated an expression to give the theorem any interest.

§22. If i is a nonion cube root of unity whose latent roots are $1, \lambda, \lambda^2$ (λ being an imaginary cube root of unity), and if ε denotes the base of the Naperian logarithms, by Sylvester's formula,

$$\varepsilon^{\theta i} = \tfrac{1}{3}\left[\varepsilon^{\theta}(i^2 + i + 1) + \lambda\varepsilon^{\lambda\theta}(i^2 + \lambda i + \lambda^2) + \lambda^2\varepsilon^{\lambda^2\theta}(i^2 + \lambda^2 i + \lambda)\right].$$

The coefficients of unity, i, and i^2, in this expression I shall denote by $f_0(\theta)$, $f_1(\theta)$ and $f_2(\theta)$; they are obviously analogous to $\sin\theta$ and $\cos\theta$ which appear in the corresponding expression for $\varepsilon^{\theta\alpha}$ in quaternions (where α is a unit vector). This gives

$$\begin{aligned} \varepsilon^{\theta i} &= f_0(\theta) + \ \ f_1(\theta).i + \ \ f_2(\theta).i^2, \\ \therefore\ \varepsilon^{\lambda\theta i} &= f_0(\theta) + \ \lambda f_1(\theta).i + \lambda^2 f_2(\theta).i^2, \\ \varepsilon^{\lambda^2\theta i} &= f_0(\theta) + \lambda^2 f_1(\theta).i + \ \lambda f_2(\theta).i^2. \end{aligned}$$

The second and third expressions are, severally, the first and second conjugates of $\varepsilon^{\theta i}$. Since $T^3\varepsilon^{\theta i} = \varepsilon^{\theta i}\varepsilon^{\lambda\theta i}\varepsilon^{\lambda^2\theta i} = 1$, hence

$$\overline{f_0(\theta)}^3 + \overline{f_1(\theta)}^3 + \overline{f_2(\theta)}^3 - 3f_0(\theta).f_1(\theta).f_2(\theta) = 1.$$

This suggests the corresponding formula $\cos^2\theta + \sin^2\theta = 1$. The properties of the sine and cosine, that $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, have their analogues in these functions; for from the values for $\varepsilon^{\lambda\theta i}$ and $\varepsilon^{\lambda^2\theta i}$, it follows that

$$f_0(\lambda\theta) = f_0(\theta),\quad f_1(\lambda\theta) = \lambda f_1(\theta),\quad f_2(\lambda\theta) = \lambda^2 f_2(\theta).$$

These functions also give rise to a formula in nonions analogous to De Moivre's theorem: thus, since $\varepsilon^{\alpha i}\varepsilon^{\beta i} = \varepsilon^{(\alpha+\beta)i}$, hence

$$\begin{aligned} &[f_0(\alpha) + f_1(\alpha).i + f_2(\alpha).i^2][f_0(\beta) + f_1(\beta).i + f_2(\beta)i^2] \\ &\qquad\qquad = f_0(\alpha+\beta) + f_1(\alpha+\beta).i + f_2(\alpha+\beta).i^2. \end{aligned}$$

Whence arise formulae for the functions of the sum of two arguments analogous to the formulae for the cosine and sine of the sum of two angles, namely,

$$\begin{aligned} f_0(\alpha+\beta) &= f_0(\alpha).f_0(\beta) + f_1(\alpha)f_2(\beta) + f_2(\alpha)f_1(\beta), \\ f_1(\alpha+\beta) &= f_0(\alpha).f_1(\beta) + f_1(\alpha).f_0(\beta) + f_2(\alpha)f_2(\beta), \\ f_2(\alpha+\beta) &= f_0(\alpha).f_2(\beta) + f_2(\alpha).f_0(\beta) + f_1(\alpha).f_1(\beta). \end{aligned}$$

It is now easy to perceive that,—just as for any quaternion q, for a proper value of θ,

$$q = \varepsilon^{\log Tq + \tan^{-1}\frac{TVq}{TSq}.\,UVq} = Tq\,(\cos\theta + \sin\theta.\,UVq);$$

so for any nonion n for proper values of θ and η,

$$\begin{aligned} n = \varepsilon^{\log Tn + \theta i + \eta i^2} &= Tn\,(f_0\theta + f_1\theta.i + f_2\theta.i^2)(f_0\eta + f_1\eta.i^2 + f_2\eta.i) \\ &= Tn\,(f_0(\theta+\eta) + f_1(\theta+\eta)i + f_2(\theta+\eta)i^2). \end{aligned}$$

We have therefore

$$\begin{aligned} n &= Tn.Un, \\ n.n' &= Tn.Tn'.Un.Un'. \end{aligned}$$

§23. Another analogy exists between quaternions and the quadrates of prime order, namely, that just as quaternions is linear in unity and two square roots of unity and their product, so the matrices of order ω constitute an algebra linear in unity and two ω^{th} roots of unity, their powers and products. Thus in the case of nonions, as Sylvester has shown,* if λ is an imaginary cube root of unity, and

$$i = \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & 1 \\ \lambda^2 & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix},$$

then $ij = \lambda^2 ji$ and $ji = \lambda ij$, while the products formed from the two sets $(1, i, i^2)$ and $(1, j, j^2)$ give all the nine units of the octanomial form of nonions. Moreover, I find that just as there are an infinite number of systems of three mutually normal unit vectors, so there is an infinite number of systems of eight cube roots of unity similar to the system formed from i and j and their products. The general expressions for these two new cube roots i_1 and j_1, from which the new system is to be formed, are

$$i_1 = (\varpi)\begin{pmatrix} 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 \end{pmatrix}(\varpi^{-1}), \quad j_1 = (\varpi)\begin{pmatrix} 0 & 0 & 1 \\ \lambda^2 & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix}(\varpi^{-1}),$$

where $|\varpi| = 0$. If the axes of i are (ρ_1, ρ_2, ρ_3) and the axes of j are $(\sigma_1, \sigma_2, \sigma_3)$, then the axes of i_1 and j_1 are respectively $(\varpi\rho_1, \varpi\rho_2, \varpi\rho_3)$ and $(\varpi\sigma_1, \varpi\sigma_2, \varpi\sigma_3)$. The condition that from i_1 and j_1 an octanomial system shall be formed similar to that formed from i and j, is that i_1 and j_1 shall have as latent roots the three cube roots of unity, and that the axes of i_1 and j_1 shall be related in the same way as those of i and j. If $(\alpha_1, \alpha_2, \alpha_3)$ represent the ground

$$\begin{aligned} \rho_1 &= \alpha_1 + \lambda\alpha_2 + \alpha_3, & \sigma_1 &= \alpha_1 + \lambda^2\alpha_2 + \alpha_3, \\ \rho_2 &= \alpha_1 + \alpha_2 + \lambda\alpha_3, & \sigma_2 &= \lambda^2\alpha_1 + \alpha_2 + \alpha_3, \\ \rho_3 &= \lambda\alpha_1 + \alpha_2 + \alpha_3, & \sigma_3 &= \alpha_1 + \alpha_2 + \lambda^2\alpha_3. \end{aligned}$$

With respect to the general case, let i be a primitive ω^{th} root of unity formed according to the method of §19, whose leading constituent is in the k^{th} place, and the coefficients of whose successive vids (beginning with that in the first row) are $1, \lambda, \lambda^2, \ldots \lambda^{\omega-1}$ (λ being an imaginary ω^{th} root of unity); and

* Johns Hopkins University Circulars, No. 17 (Aug. 1882).

let j be formed from the same vids with the coefficients of i replaced respectively by their squares; then $ij = \lambda^{\omega-k+1}ji$ and $ji = \lambda^{k-1}ij$. If $k = 2$, $ij = \lambda^{\omega-1}ji$ and $ji = \lambda ij$.

Quadrate Algebras whose Order is not a Prime Number.

§24. With regard to the quadrates whose order is not a prime number, it may be shown also by a method similar to the method of §19, that they possess $\omega^2 - 1$ linearly independent non-scalar ω^{th} roots of unity, formed from their vids in the same manner in which the i, j, k of quaternions are formed from the vids of a matrix of order two. However, of these $\omega^2 - 1$ ω^{th} roots, only $(\omega + 1).\tau(\omega)$ (where $\tau(\omega)$ denotes the totient of ω) are primitive, namely, $\tau(\omega)$ ω^{th} roots formed from the vids along the diagonal, and $\omega.\tau(\omega)$ others formed from the non-symmetric vids. The latter consist of those roots in which the leading constituent is in a row whose order less one is prime to ω. The roots in which the leading constituent is in a row whose order less one has with ω the greatest common divisor ∂, other than unity, are either $(\omega : \partial)^{\text{th}}$ roots of unity, or such to a factor *près*. Of the ω^{th} roots of unity formed from the vids along the diagonal by taking as their coefficients powers of the scalar ω^{th} roots of unity, $1, \lambda, \lambda^2, \ldots . \lambda^{\omega-1}$, only those are primitive ω^{th} roots of which the power taken of the set of λ's is an integer prime to ω.

However, all these quadrates are analogous to quaternions in admitting of selective symbols, in having functions analogous to the conjugate in quaternions, and in that any expression in them is resolvable into the product of a tensor and a versor. This may be proved in precisely the same way in which in §21 and §22 it is shown to be true for the quadrate algebra of order three.

The quadrate algebras whose order is not a prime number are compounds of other algebras, by which I mean that they are linear in the products of the units of these algebras: if $\omega = \omega'.\omega''$, then the quadrate algebra of order ω is linear in the products of the units of the quadrate algebras of order ω' and ω'', the units of each of these systems being regarded as commutative with those of the other. Thus, the algebra formed from the vids of a matrix of order four is linear in the products of two quaternion sets, the i, j, k of the one set being commutative with the i, j, k of the other set; the algebra formed from the vids of a matrix of order six is a compound of a quaternion set and a nonion set, the units of the quaternion set being commutative with those of the nonion

set. It is obviously sufficient to prove this for any form of the several algebras, and for this purpose I choose the canonical form, and I shall show that the vids of any matrix of order ω' combined with those of any matrix of order ω'' (the vids of the first set being regarded as commutative with those of the second set), give an algebra whose multiplication table is the same as that formed from the vids of a matrix of order $\omega = \omega'.\omega''$. I shall, however, first illustrate this theorem by the case of the matrix of order four. Considering the two quadrate systems

$$\begin{array}{cccc} (A_1:A_1) & (A_1:A_2) & (B_1:B_1) & (B_1:B_2), \\ (A_2:A_1) & (A_2:A_2) & (B_2:B_1) & (B_2:B_2), \end{array}$$

which will be regarded as commutative, evidently the complete system formed by their products will consist of sixteen linearly independent units which may be arranged as follows:

$$\begin{array}{cc|cc} (A_1:A_1)(B_1:B_1) & (A_1:A_2)(B_1:B_1) & (A_1:A_1)(B_1:B_2) & (A_1:A_2)(B_1:B_2) \\ (A_2:A_1)(B_1:B_1) & (A_2:A_2)(B_1:B_1) & (A_2:A_1)(B_1:B_2) & (A_2:A_2)(B_1:B_2) \\ \hline (A_1:A_1)(B_2:B_1) & (A_1:A_2)(B_2:B_1) & (A_1:A_1)(B_2:B_2) & (A_1:A_2)(B_2:B_2) \\ (A_2:A_1)(B_2:B_1) & (A_2:A_2)(B_2:B_1) & (A_2:A_1)(B_2:B_2) & (A_2:A_2)(B_2:B_2) \end{array}$$

Those compound vids in the upper left-hand group consist of the first set, each multiplied by $(B_1:B_1)$; those in the other groups also of the first set multiplied respectively by $(B_1:B_2)$, $(B_2:B_1)$ and $(B_2:B_2)$. According to this scheme the product of the vid $(A_x:A_y)$ from the first system and the vid $(B_r:B_t)$ in the second system occupies in the resulting system a position which may be denoted by $(x + \omega'\overline{r-1},\ y + \omega'\overline{t-1})$, where, as before, the first number denotes the row and the second the column; and in this case $\omega' = \omega'' = 2$. On trial it will be found that this compound system has the same multiplication table as that of the system of vids of a matrix of order four, the vids in the compound system corresponding to those of the quadrate system of order four which occupy the same place. It should be observed that if I and I' are vids of the first quadrate system and J and J' vids of the second, that the product of the compound vids $I.J$ and $I'.J'$ is zero, unless $I.I' \neq 0$ and $J.J' \neq 0$; since the I's and J's are commutative. The proof in the general case may be accomplished as follows: Following the same scheme of arrangement of the compound vids formed from

the product of the vids of two quadrate systems of order ω' and ω'' respectively, we have

$$IJ \equiv (A_x : A_y)(B_r : B_s) = (x + \omega' \overline{r-1},\ y + \omega' \overline{s-1}) \equiv K,$$
$$I'J' \equiv (A_{y'} : A_z)(B_{s'} : B_t) = (y' + \omega' \overline{s'-1},\ z + \omega' \overline{t-1}) \equiv K'.$$

The product of IJ and $I'J'$ is zero, unless $y' = y$ and $s' = s$, when we have

$$IJ . I'J' = (A_x : A_z)(B_r : B_t),$$

which in the compound system will occupy the $(x + \omega' \overline{r-1})^{\text{th}}$ row and $(z + \omega' \overline{t-1})^{\text{th}}$ column, and thus is represented by $(x + \omega' \overline{r-1},\ z + \omega' \overline{t-1})$. The necessary and sufficient condition, however, that the compound system shall have the same law of multiplication as that of the algebra formed from the vids of a matrix of order $\omega = \omega' . \omega''$ is that KK' shall be zero unless

$$y' + \omega' \overline{s'-1} = y + \omega' \overline{s-1},$$

in which case we must have

$$KK' = (x + \omega' \overline{r-1},\ z + \omega' \overline{t-1}).$$

But since $y' \leqq \omega'$, $y \leqq \omega'$, if $y' + \omega' \overline{s'-1} = y + \omega' \overline{s-1}$, then $y' = y$ and $s' = s$. Hence the condition is fulfilled.

§25. From the last section it follows that the matrix of order $\omega = 2^m$ is a compound of m quaternion algebras which do not interfere, i. e. the units of each set are commutative with those of the other sets. I shall term such an algebra the *m-way quaternion algebra.* The system of m-way quaternion algebras has already been considered by Clifford (this Journal, Vol. I). Clifford, however, approached the subject from an entirely different point of view. He starts with n "elementary units" $i_1 i_2 \ldots i_n$ whose multiplication with each other is polar, and which are such that the square of each is -1. The 2^n products, of order 0 to n, of these elementary units, are linearly independent; and Clifford shows that the products of even order, 2^{n-1} in number (which may be obtained from all combinations of the binary products $i_1 i_2$, $i_1 i_3$, etc), form an algebra or system by themselves, which he terms the *n-way algebra.* If $n = 2m + 1$, Clifford further proves that "the n-way algebra is a compound of m sets of quaternions which do

not interfere;" and if $n = 2m$, he shows that the n-way algebra is a compound of m quaternion sets which do not interfere, and the algebra $(1, \varepsilon)$, where $\varepsilon^2 = 1$ and ε is commutative with each of the quaternion sets. At present we are only concerned with the n-way algebra when n is odd; it consists, we have seen, of $2^{n-1} = 2^{2m}$ linearly independent units, and is a compound of m quaternion sets which do not interfere. Hence Clifford's $n (= 2m + 1)$-way algebra is that formed from the vids of a matrix of order 2^m, or is the m-way quaternion algebra. Commenting on the surprising fact that sets of quaternions should appear as the simplest form of an algebra which at first sight is so far from suggesting Hamilton's system, Clifford says that "thus it appears that quaternions is the last word of algebra to geometry." It is still more surprising that quaternions should be so prominent in the theory of matrices, and in a sense embrace the whole subject; for since the theory of matrices of any order is included in the theory of matrices of higher order, and since however great a number may be, a power of two may be obtained which is still greater, it follows that the theory of matrices of any order is included in the theory of the combination of a certain finite number of quaternion sets which do not interfere.

With regard to Clifford's geometrical algebras I am able to show that the entire system formed from the combinations of all orders of the n elementary units, which I term the n-fold algebra, is identical with the $(n-1)$-way algebra; and when $n = 2m$, by an argument similar to that employed in the case of bi-quaternions by Benjamin Peirce, that the n-way algebra is a compound of two groups of m sets of quaternions which do not interfere, such that every set of either group is nil-facient and nil-faciend with any set of the second group.

It is clear that any quantity in Clifford's n-fold algebra may be expressed as a sum of products of expressions $\alpha = \Sigma a_1 i_1$, $\beta = \Sigma b_1 i_1$, etc., linear in the elementary units; the α, β, etc., are of course such that $\alpha^2 = -\Sigma a_1^2$, $\beta^2 = -\Sigma b_1^2$; whence, on account of the obvious analogy, they may be termed vectors. In virtue of the relation stated above which a matrix of any order holds to those of higher order, and the consequent inclusion of the theory of matrices of any order in that of sets of commutative quaternions, the theorem follows that the theory of matrices is the theory of expressions which are sums of products of quantities α, β, etc., whose squares are negative scalars.